Divide knot presentation of Berge’s knots of lens space surgery

Yuichi YAMADA  (Univ. of Electro-Comm. Tokyo)

Singularities, knots, and mapping class groups in memory of Bernard Perron

2010 Sept.
Univ. of Bourgogne, Dijon, France
1. Introduction. Dehn surgery

2. Divide knots

3. Lens space surgery

4. Results. (Divide presentation of Berge knots)
§1. Dehn surgery

Dehn surgery = Cut and paste of a solid torus.

\[(K; p) := (S^3 \setminus \text{open nbd} \mathcal{N}(K)) \cup_{\partial} \text{Solid torus}.\]

Coefficient (in \(\mathbb{Z}\)) “framing” = a parallel curve (\(\subset \partial \mathcal{N}(K)\)) of \(K\), or the linking number.

Solid torus is reglued such as “the meridian comes to the parallel”
**Theorem ([Lickorish ’62])**

Any closed connected oriented 3-manifold $M$ is obtained by a framed link $(L, p)$ in $S^3$, ie, $M = (L; p)$,

$$(L, p) = (K_1, p_1) \cup (K_2, p_2) \cup \cdots \cup (K_n, p_n).$$

**Lens space** $L(p, q)$

\[
\frac{p}{q} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots - \frac{1}{a_n}}}
\]

$(a_i > 1)$

\[
-\frac{p}{q} = -a_1 - a_2 - a_3 - \cdots - a_n
\]

For $n \in \mathbb{Z}$, $r \in \mathbb{Q}$
Theorem (Kirby-Rolfsen moves (Fenn-Rourke’s ver.))

The 3-manifolds are homeo. \((L; p) \cong (L'; p')\)

\(\Leftrightarrow\) framed links \((L, p), (L', p')\) are moved to each other by isotopy and the following

Note: This (with a suitable sign) is blow-down/up, related to resolution of the singularity.

The green curve (axis of the twist) is called the exceptional curve.
Starting example of lens space surgery [Fintushel–Stern ’80]
It is not easy to find/prove “unexpected” lens space surgery.

\[ P(-2, 3, 7) \]

18–surgery is
\[-2 \quad -3 \quad -4 \]

\[ ? \]

19–surgery is
\[-2 \quad -3 \quad -2 \quad -3 \]

What is the best method to prove it?
My answer ([Y]):

\[ P(-2, 3, 7) \]

\[ S^3 \]

blue \cup green = S^3, and red becomes the knot \( P(-2, 3, 7) \).

The knot is constructed by seq. of full-twists = blow-downs.
Introduction. Dehn surgery

Divide knots

Lens space surgery

Results. (Divide presentation of Berge knots)

Knots
are constructed
by some full-twists

$\iff$

Singularity
are resolved
by some blow-ups

In 2000, I heard A’Campo’s divide theory:

\[
\text{a generic plane Curve } P \quad \Rightarrow \quad \text{a Link } L(P) \text{ in } S^3
\]

A’Campo’s divide knots [’75]

— Plane curves are easier to draw than knots.
Typical example of my results.

Area = 18

Area = 19

18-surgery is $L(18, -7)$

$P(-2, 3, 7)$

19-surgery is $L(19, -7)$
Theorem (Main Theorem)

Every Berge’s knot of lens space surgery is a divide knot.

Berge’s list (’90), is believed to be the complete list of lens space surgery.

For the proof, it took many years. Because I am lazy, but

As a phenomenon, lens space surgery is not simple.
The set (list) consists of three subfamilies of infinite knots. Each subfamily has each “personality”.
At first, I hoped to synthesize all of them deductively.
But after all, Each subfamily needs each method in the detail.
By another approach (Heegard Floer homology, \( \mathbb{C} \)-links by Rudolph \( \cdots \)), it is proved:

**Lemma (Hedden)**

*Any knots of lens space surgery is intersection of an algebraic surface in \( \mathbb{C}^2 \) and a 4-ball.*

My study is more concrete, to know
- “How” does each knot yield a lens space?
- the construction of *each knot* of lens space surgery,
- the set of lens space surgeries. \( \cdots \)
§2. A’Campo’s divide knots

Original construction.

\[
\begin{align*}
\text{a generic plane Curve } P & \quad \Rightarrow \quad \text{a Link } L(P) \text{ in } S^3 \\
& \quad \text{A’Campo’s divide knots}
\end{align*}
\]

Let \( P \) be a generic (no self-tangency) curve in the unit disk \( D \),

\[
S^3 = \{(u, v) \in TD | u \in D, v \in T_uD, |u|^2 + |v|^2 = 1\}
\]

\[
L(P) := \{(u, v) \in TD | u \in P, v \in T_uP, |u|^2 + |v|^2 = 1\} \subset S^3.
\]

\[
\text{Here is the strongly-involution } \iota : (u, v) \mapsto (u, -v).
\]

\[
\text{Diagram:}
\]

\[
\text{Diagram:}
\]
**Ex. Torus links** [’02 Goda-Hirasawa-Y, (Gusein-Zade, etc.)]

$P$ is the $p \times q$ rectangle (billiard) curve $\Rightarrow L(P)$ is $T(p, q)$

(PL curve with slope $\pm 1$ in the rectangle.)

ex. $(p, q) = (7, 4)$

It has $\frac{(p - 1)(q - 1)}{2}$ double points, in general.
Basics on Divide knots [N.A’Campo, L.Rudolph,..]

(0) $L(P)$ is a knot ($\#L(P) = 1$) $\iff$ $P$ is an immersed arc.

(1) The genus of knot $L(P) = \#$ double points of $P$.

(2) $\text{lk}(L(P_1), L(P_2)) = \#(P_1 \cap P_2)$, if they are knots.

(3) Every divide knot $L(P)$ is fibered.

(4) Any divide knot is a closure of strongly quasi-positive braid.

ie, product of some $\sigma_{ij}$.

(5) $P_1 \sim P_2$ by $\Delta$-move $\implies$ $L(P_1) = L(P_2)$. $L$ is not injective.

\[ \Delta \text{-move on plane curves} \]
These curves present the same knot $P(-2, 3, 7)$
(Thanks to Hirasawa)

Only 19 and 18 are the coefficients of lens space surgery.
“Ordered Morse divide (OMD)” [Couture]

A divide is called \textit{OMD} if (w.r.t at least one direction) max/min points are in the same level, up to isotopy.

\[ \rightarrow This \ is \ a \ Non-OMD. \]
\( (w.r.t \ horizontal \ nor \ vertical \ direction) \)
’01 Couture-Perron proved a visualized version

\[
\text{an OMD } P \quad \Rightarrow \quad \text{a knot } L(P) \text{ in } S^3
\]

Braid presentation of \( L(P) \)

In this talk, we demonstrate them by \( P(-2, 3, 7) \)
[’01 Couture-Perron] proved a visualized version

\[
\text{an OMD } P \quad \Rightarrow \quad \text{a knot } L(P) \text{ in } S^3
\]

Braid presentation of \( L(P) \)

The braid presentation of \( L(P) \).

We get the fiber surface of \( L(P) \).  genus = 5.
The result is $W_4^3 W_3^2$, where $W_n := \sigma_{n-1} \sigma_{n-2} \cdots \sigma_2 \sigma_1$ “1/n twist”.

**Exercise.** Show that $W_5^3 W_3$ induces the same knot.
We focus on curves cut from the lattice

Assume (on the regions)
- Vertices are in $\mathbb{Z}^2 \subset \mathbb{R}^2$,
- Edges of the region are vertical or horizontal,
- Every concave point is at odd point, and
- The curve $P$ is an immersed arc ($\iff L(P)$ is a knot).

where we call a point $(m, n) \in \mathbb{Z}^2$ odd if $m + n \equiv 1 \mod 2$

odd. OK

even. Not allowed
Examples.
In fact, these examples present knots of lens space surgery.

- L-shaped curve

- Generalized L-shaped curve

- T-shaped curve

- Generalized T-shaped curve
Lemma (Y)

"Adding a square" corresponds to a right-handed full-twist = blow-down = coord. transform: \((x, y) = (X, Y/X)\).
(example: \(y^2 = x + \epsilon\) becomes \(Y^2 = X^2(X + \epsilon)\))
**Cable knot:** $C(T(a, b); p, pab + r)$ of a torus knot is a divide knot.

ex. $C(T(2, 3); 2, 13)$

It is algebraic: From $T(a, b) = \begin{cases} x = t^a \\ y = t^b \end{cases}$ to $\begin{cases} x = t^{ap} \\ y = t^{bp} + t^{bp+r} \end{cases}$, or

$$y = x^b a \left(1 + x^r \frac{a}{ap}\right)$$

Puiseux pair is $\{(b, a), (bp + r, p)\}$. 
More iterated cables (⇒ generalized L-shaped)

\[ C( C( T(a, b); p, pab + r); P, Pp(pab + r) + R) \] is represented,
(ex. \[ C( C( T(2, 3); 2, 13); 3, 80) \],
\[ y = x \frac{b}{a} \left( 1 + x \frac{r}{ap} \left( 1 + x \frac{R}{apP} \right) \right) \].

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]

\[ \text{Diagram:} \]
Application: Twisted torus knot

$T(p, q; r, s)$ is constructed from $T(p, q)$ (in the standard position) by $s$ full-twists of the $r$ strings in the $p$ strings.

Fact. $T(p, q; r, s)$ can be cable knots, for specially controlled $p, q, r, s$ ([Morimoto-Y], SangYop Lee)
§3. Lens space surgery

"Which \((K; p)\) is a lens space?" \(K\): a knot

ex.1 ['71 L. Moser] Torus knots.

\[ p = ab \pm 1 \Rightarrow (T(a, b); p) \cong L(p, -b^2). \]

\[ K := T(3, 5), \text{ then } (K; 16) = L(16, 7) \text{ and } (K; 14) = L(14, 5). \]

ex.2 ['77 J. Bailey, D. Rolfsen] 2 Cables of Torus knots

— Shown in §2. —

ex.3 ['80 R. Fintushel, R. Stern] Hyperbolic knot!

\[ K := P(-2, 3, 7), \text{ then } (K; 19) = -L(19, 7). \]

\[ (K; 18) = -L(18, 7). \]
Berge’s doubly-primitive knots [’90]
A knot $K$ in the Heegaard surface $\Sigma_2$ is *doubly-primitive* iff

$$K_\# \text{ (as in } \pi_1) \text{ is a generator in both } \pi_1(H_2) \text{ and } \pi_1(-H_2).$$

Such a knot $K$ with the surface slope (coeff.) always yields a lens space. ■

Berge (tried to) classified and made a list of such knots. His list consists of 3 Subfamilies (and of 12 “Type”s).

<p>| Type I, II, III, …, VI | VII, VIII | IX, …, XII |</p>
<table>
<thead>
<tr>
<th>Doubly Primitive knots</th>
<th>Torus knots</th>
<th>2cables of torus knots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid Torus [91]</td>
<td><em>P18</em></td>
<td></td>
</tr>
<tr>
<td>g=1 fiber surface</td>
<td><em>P19</em></td>
<td></td>
</tr>
<tr>
<td>Sporadic</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Torus knots**: *P18*
- **2cables of torus knots**: *P19*
- **Trefoil fib.surf.**: fig8 fib.surf.

**Notes**:
- Introduction. Dehn surgery
- Divide knots
- Lens space surgery
- Results. (Divide presentation of Berge knots)
- Gordon conj.

**Knots of lens surgery**
Berge’s list (Subfamilies)

(1) Knots in the solid torus (Berge-Gabai knots)
   Type I : Torus knots
   Type II : 2-cables of torus knots
   Type III · · · VI : “generic” lens surgery

(2) Knots in the genus 1 fiber surface
   Type VII : knots in the trefoil fiber surface $F^+$
   Type VIII : knots in the fig8. fiber surface $F^-$

(3) Sporadic examples
   Type IX · · · XII : Sporadic knots
   (It is known XI = X, XII = IX)


§4. Results

**Results**: Divide presentation of Berge’s knots
Theorem (1). ([Y ('09 AGT)])]
Every knot (up to mirror image)
in the subfamily (1), Type I, II, III, ... VI,
- is a **divide knot**,
- is presented by an **L-shaped plane curve** s.t.

\[ \text{Area}(L) - \text{coeff.} = 0 \text{ or } 1. \]

ex. Type III knots are parametrized by

\[ \delta, \varepsilon \in \{\pm 1\}, \ A(\geq 2), \ k(\geq 0) \text{ and } t \in \mathbb{Z}. \]

Thus we call the knots

\[ k_{\text{III}}(\delta, \varepsilon, A, k, t). \]
Type III knot: \( k_{III}(-1, +1, A, k, t) \)

**ex.** \( P(-2, 3, 7) \) (with 18-surgery) is \( k_{III}(-1, +1, 2, 0, 0) \), ie, \( A = 2, k = t = 0 \). The divide presentation is

Green lines are used for \( k, t > 0 \), we **add squares** (full-twists).

Notation:
**Theorem (2-1).** ([Y ’05 JKTR])

**TypeVII** (L-shaped)

Let $F^+$ be the fiber surface of the left-handed trefoil.

- [Y’10] Any TypeVII knot is $k^+(a, b)$ in $F^+$ with a positive coprime $(a, b)$ s.t. $0 < a < b$.

Its $p$-surgery is $L(p, q)$.

$(p = a^2 + ab + b^2, q = -(a/b)^2 \mod p)$

- $k^+(2, 3)$ is $P(-2, 3, 7)$. $2^2 + 2 \cdot 3 + 3^2 = 19$. 
Theorem (2-2).

**Type VIII**  The most difficult.

Let $F^-$ be the fiber surface of the 8-knot.

- [Y’10] Any TypeVIII knot is $k^-(a, b)$, with a positive coprime $(a, b)$ s.t. $0 < a < b/2$.

Its $p$-surgery is $L(p, q)$.

$$(p = -a^2 + ab + b^2, q = -(a/b)^2 \mod p)$$

- It is known $k^-(a, b) = k^-(b - a, b)$.

- $k^-(2, 5) = -2^2 + 2 \cdot 5 + 5^2 = 31$.

On divide knots, negative twists are hard to treat with.
\( k^-(a, b) \) is a divide knot. The plane curve is constructed by a blow-down from the rectangle curve \( a \times (b - a) \), as follows:

**ex.** \( k^-(2, 5) \) is from \( T(2, 3) \)

Let’s do the blow-down.

**Key** of the proof was [’05 Baker]’s deformation:
Blow-down from $T(2, 3)$ to construct $k^-(2, 5)$.

If $a, b$ are large, it is a hard homework.
**Type VIII knots.**

- $k^-(2, b)$
  - Area: $b^2 + 2b - 2$
  - Coeff.: $b^2 + 2b - 4$
  - Area: 64
  - Coeff.: 61

- $k^-(3, 7)$
  - Area: $b^2 + 2b - 2$
  - Coeff.: $b^2 + 2b - 4$
  - Area: 82
  - Coeff.: 79

- $k^-(3, 8)$
  - Area: $b^2 + 2b - 2$
  - Coeff.: $b^2 + 2b - 4$
  - Area: 82
  - Coeff.: 79
**Theorem (3). Type IX and X**

- ([’05 Baker])

The knot $k_{IX}(j)$ and $k_X(j)$, ($j \in \mathbb{Z}$)

is obtained from $T(j, j + 1)$ by full-twists three times

for Type IX, in order (a → b → c), $p = 22j^2 + 9j + 1$,

for Type X, in order (a → c → b), $p = 22j^2 + 13j + 2$. 

\(j=2\) case

![Diagram of the knot and its diagram presentation](image-url)
**Sporadic knots in Type IX.** \((p = 22j^2 + 9j + 1)\) and **Type X.**

- \(k_{\text{IX}}(1)\), L-shaped, \(p = 32\), Area=33.
- \(k_{\text{IX}}(2)\), generalized L-shaped, \(p = 107\), Area=109.

**Fact.** (maybe known to experts) \(k^{-}(2, 5)\) (in **Type VIII**, \(p = 31\)) is the same knot \(k_{\text{IX}}(1)\) (\(p = 32\)), but the coefficients \(p\) differ.
My work is not completed yet in the following sense:

**Q1.** Decide the final, best divide presentation, for every knots (Type \textbf{VIII}) in Berge’s list of lens space surgery.

— minimal area, good-shaped, or...? —

The presentation is not unique, because of at least $\Delta$-moves.

Related question:

**Q2.** For a given plane curve, determine whether it presents a knot of lens space surgery or not, and if it is, get the coefficient $p$ (and Berge’s parameter) from the curve.

**Q3.** Find more exceptional Dehn surgery along divide knots.

- ’09 Couture] introduced Khovanov “Categorification” invariants for divide knots...
Thank you very much!