Divide knot presentation of Berge's knots of lens space surgery

Yuichi YAMADA (Univ. of Electro-Comm. Tokyo)

Singularities, knots, and mapping class groups
in memory of Bernard Perron

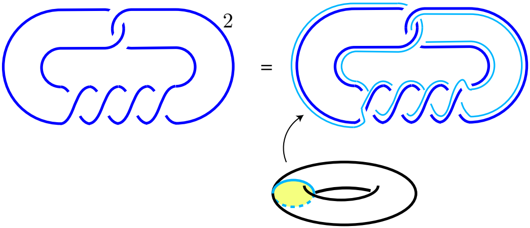
2010 Sept.
Univ. of Bourgogne, Dijon, France

§1. Dehn surgery

Dehn surgery $=$ Cut and paste of a soliod torus.

$$(K;p) := \left(S^3 \setminus \mathrm{open\ nbd} N(K)\right) \cup_\partial \mathrm{Solid\ torus.}$$

Coefficient (in $\mathbf{Z}$) "framing" $=$ a *parallel* curve ($\subset \partial N(K)$) of K,
or the linking number.
Solid torus is reglued such as "the meridian comes to the parallel"



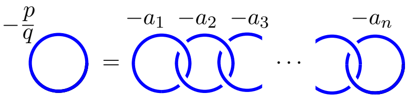
Theorem ([Lickorish '62])

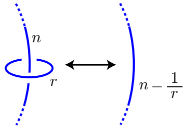
Any closed connected oriented 3-manifold M is obtained by a framed link $(L, \mathbf{p})$ in S^3, ie, $M = (L; \mathbf{p})$,

$$(L, \mathbf{p}) \;=\; (K_1, p_1) \cup (K_2, p_2) \cup \cdots \cup (K_n, p_n).$$

Lens space $L(p, q)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$

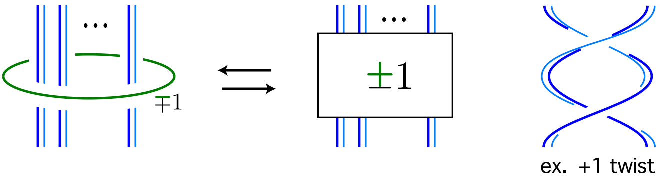




For $n \in \mathbf{Z}, r \in \mathbf{Q}$

Theorem (Kirby-Rolfsen moves (Fenn-Rourke's ver.))

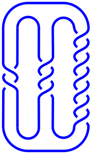
The 3-manifolds are homeo. $(L;\mathbf{p}) \cong (L';\mathbf{p}')$
$\Leftrightarrow$ *framed links* $(L,\mathbf{p}),(L',\mathbf{p}')$ *are moved to each other by isotopy and the following*



Note: This (with a suitable sign) is blow-down/up,
related to resolution of the singularity.
The green curve (axis of the twist) is called the exceptional curve.

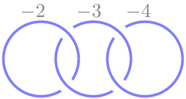
Starting example of lens space surgery [Fintushel–Stern '80]

It is not easy to find/prove "unexpected" lens space surgery.

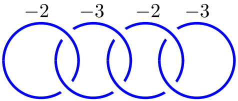


$P(-2,3,7)$

18–surgery is



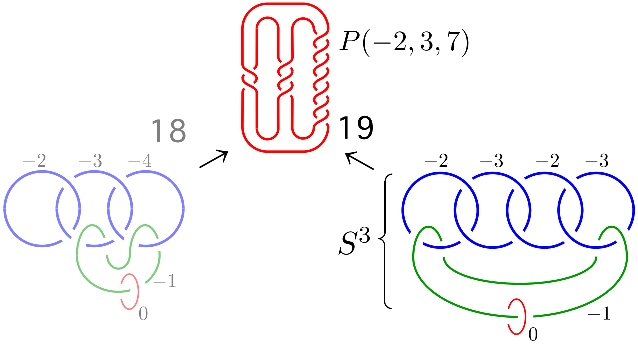
19–surgery is



?

What is *the best method* to prove it?

My answer ([Y]) :



blue ∪ green = S^3, and red becomes the knot $P(-2, 3, 7)$.
The knot is constructed by seq. of full-twists = blow-downs.

Knots are constructed by some full-twists	$\Leftrightarrow$	**Singularities** are reslolved by some blow-ups

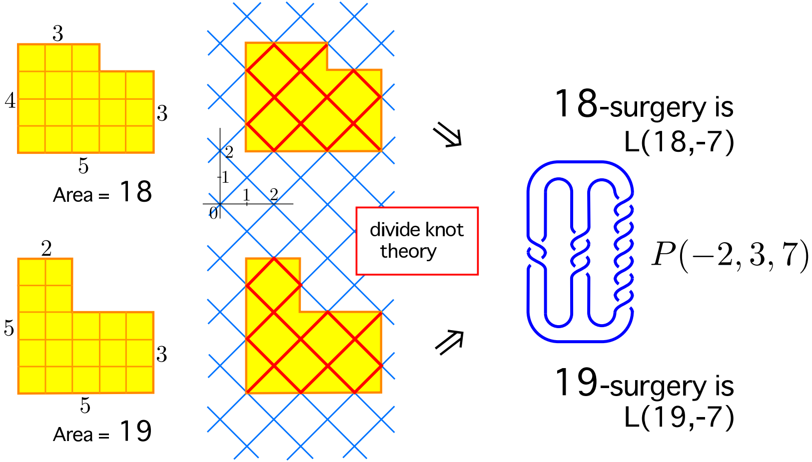
In 2000, I heard A'Campo's divide theory:

a generic plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

A'Campo's divide knots ['75]

—— Plane curves are easier to draw than knots.

Typical example of my results.



Theorem (Main Theorem)

Every Berge's knot of lens space surgery is a divide knot.

Berge's list ('90), is believed to be the complete list of lens space surgery.

For the proof, it took many years.

Because I am lazy, but

As a phenomenon, lens space surgery is not simple.
The set (list) consists of **three subfamilies** of inifinite knots. Each subfamily has each "personality".
At first, I hoped to synthesize all of them deductively.
But after all, *Each subfamily needs each method in the detail*.

By another approach (Heegarrd Floer homology, $\mathbb{C}$-links by Rudolph $\cdots$), it is proved:

Lemma (Hedden)

Any knots of lens space surgery is intersection of an algebraic surface in $\mathbb{C}^2$ *and a 4-ball.*

My study is more concrete, to know

· "How" does each knot yield a lens space?

· the construction of *each knot* of lens space surgery,

· the set of lens space surgeries. $\cdots$

§2. A'Campo's divide knots

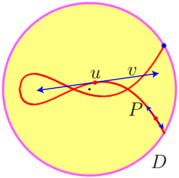
Original construction.

a generic plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

A'Campo's divide knots

Let P be a generic (no self-tangency) curve in the unit disk D,

$$S^3 = \{(u, v) \in TD | u \in D,\ v \in T_uD,\ |u|^2 + |v|^2 = 1\}$$
$$L(P) := \{(u, v) \in TD | u \in P,\ v \in T_uP,\ |u|^2 + |v|^2 = 1\} \subset S^3.$$

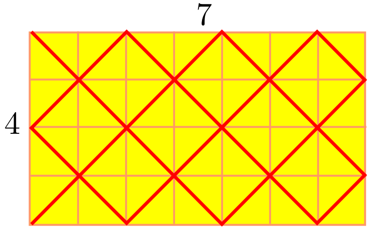


· Here is the *strongly-involution* $\iota : (u, v) \mapsto (u, -v)$.

Ex. Torus links ['02 Goda-Hirasawa-Y, (Gusein-Zade, etc.)]

P is the $p \times q$ rectangle (billiard) curve $\Rightarrow$ $L(P)$ is $T(p, q)$
(PL curve with slope ± 1 in the rectangle.)

ex. $(p, q) = (7, 4)$



It has $\dfrac{(p-1)(q-1)}{2}$ double points, in general.

Basics on Divide knots [N.A'Campo, L.Rudolph,..]

(0) $L(P)$ is a knot ($\sharp L(P) = 1$) $\Leftrightarrow$ P is an immersed arc.

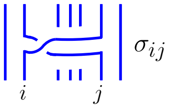
(1) The genus of knot $L(P)$ = $\sharp$ double points of P.

(2) $\mathrm{lk}(L(P_1), L(P_2)) = \sharp(P_1 \cap P_2)$, if they are knots.

(3) Every divide knot $L(P)$ is *fibered*.

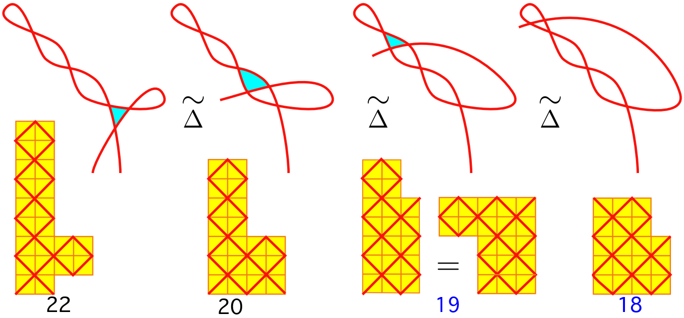
(4) Any divide knot is a closure of *strongly quasi-positive* braid.
ie, product of some σ_{ij}.

(5) $P_1 \sim P_2$ by Δ-move $\Rightarrow$ $L(P_1) = L(P_2)$. L is *not injective*.



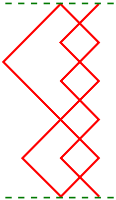
Δ-move on plane curves

These curves present the same knot $P(-2, 3, 7)$
(Thanks to Hirasawa)

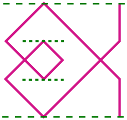


Only 19 and 18 are the coefficients of lens space surgery.

"Ordered Morse divide (OMD)" [Couture]



A divide is called *OMD*
if (w.r.t at least one direction) max/min points are in the same level, up to isotopy.



⇐ This is a Non-OMD.
(w.r.t horizontal nor vertical direction)

['01 Couture-Perron] proved a visualized version

an OMD P $\Rightarrow$ a knot $L(P)$ in S^3

Braid presentation of $L(P)$

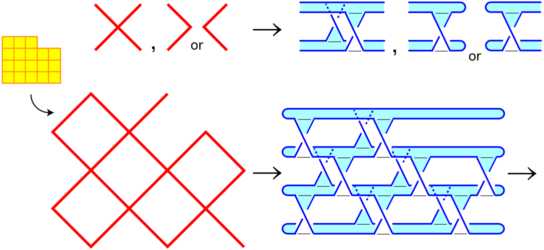
In this talk, we demonstrate them by $P(-2,3,7)$

['01 Couture-Perron] proved a visualized version

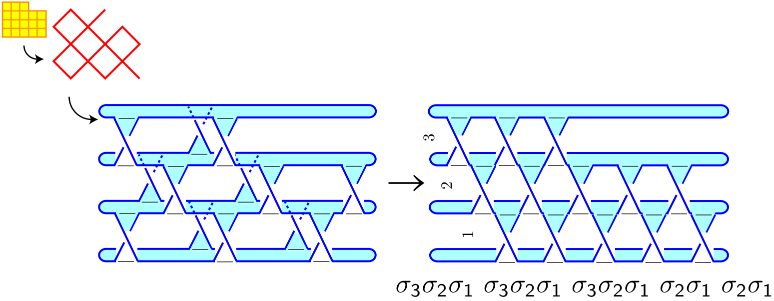
an OMD P $\Rightarrow$ a knot $L(P)$ in S^3

Braid presentation of $L(P)$

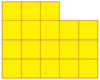
The braid presentation of $L(P)$.



We get the fiber surface of $L(P)$. genus $= 5$.



The result is ${W_4}^3 {W_3}^2$, where
$W_n := \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1$ "$1/n$ twist".

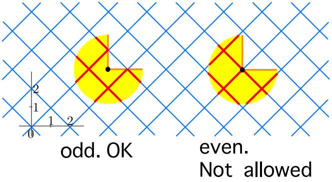


Exercise. Show that ${W_5}^3 W_3$ induces the same knot.

We focus on curves cut from the lattice

Assume (on the regions)
- Vertices are in $\mathbf{Z}^2 \subset \mathbf{R}^2$,
- Edges of the region are vertical or horizontal,
- Every concave point is at odd point, and
- The curve P is an immersed arc ($\Leftrightarrow L(P)$ is a knot).

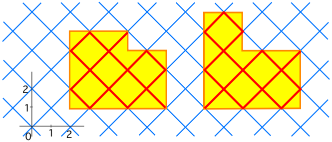
where we call a point $(m, n) \in \mathbf{Z}^2$ odd if $m + n \equiv 1 \mod 2$



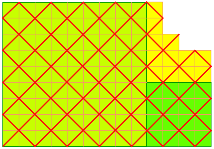
Examples.

In fact, these examples presents knots of lens space surgery.

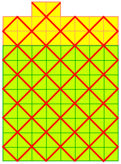
· L-shaped curve



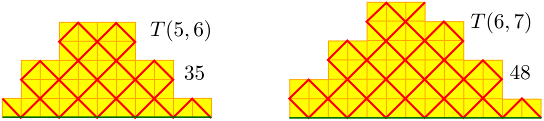
· Generalized L-shaped curve



· T-shaped curve

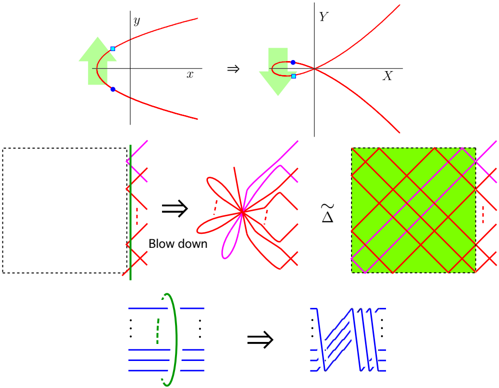


· Generalized T-shaped curve

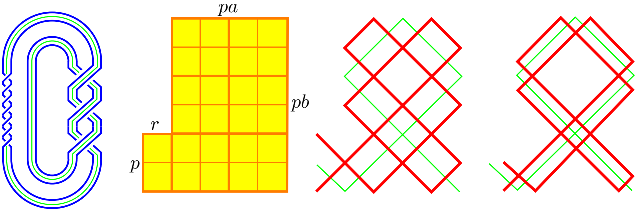


Lemma (Y)

“**Adding a square**” *corresponds to a right-handed* **full-twist** $=$ **blow-down** $=$ *coord. transform:* $(x, y) = (X, Y/X)$.
(ex. $y^2 = x + \epsilon$ *becomes* $Y^2 = X^2(X + \epsilon)$*)*



Cable knot: $C(T(a,b);p,pab+r)$ of a torus knot is a divide knot.
ex. $C(T(2,3);2,13)$

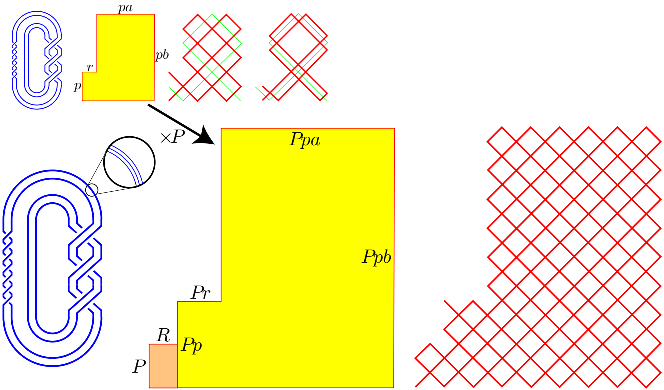


It is algebraic: From $T(a,b)=\begin{cases}x=t^a\\ y=t^b\end{cases}$ to $\begin{cases}x=t^{ap}\\ y=t^{bp}+t^{bp+r}\end{cases}$, or

$$y=x^{\frac{b}{a}}\left(1+x^{\frac{r}{ap}}\right)$$

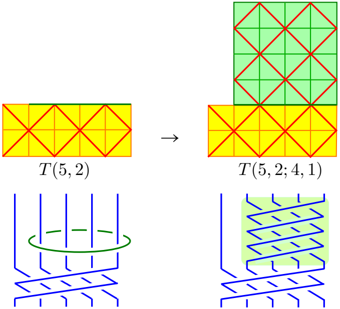
Puiseux pair is $\{(b,a),(bp+r,p)\}$.

More iterated cables (⇒ generalized L-shaped)
$C(\,C(T(a,b);p,pab+r);P,Pp(pab+r)+R))$ is represented,
(ex. $C(\,C(T(2,3);2,13);3,80))$, $y=x^{\frac{b}{a}}\left(1+x^{\frac{r}{ap}}\left(1+x^{\frac{R}{apP}}\right)\right)$.



Appplication : Twisted torus knot

$T(p, q; r, s)$ is constructed from $T(p, q)$ (in the standard position) by s full-twists of the r strings in the p strings.

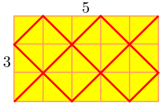


Fact. $T(p, q; r, s)$ can be cable knots, for specially controlled p, q, r, s ([Morimoto-Y], SangYop Lee)

§3. Lens space surgery

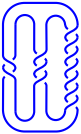
"Which $(K;p)$ is a lens space?" K: a knot

ex.1 ['71 L. Moser] **Torus knots.**
$p = ab \pm 1 \Rightarrow (T(a,b);p) \cong L(p,-b^2)$.
$K := T(3,5)$, then $(K;16) = L(16,7)$ and $(K;14) = L(14,5)$.



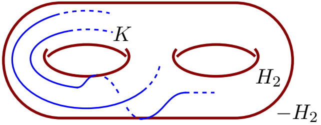
ex.2 ['77 J. Bailey, D. Rolfsen] **2 Cables of Torus knots**
— Shown in §2. —

ex.3 ['80 R. Fintushel, R. Stern] Hyperbolic knot!
$K := P(-2,3,7)$, then $(K;19) = -L(19,7)$.
$(K;18) = -L(18,7)$.



Berge's doubly-primitive knots ['90]
A knot K in the Heegaard surface Σ_2 is *doubly-primitive* iff

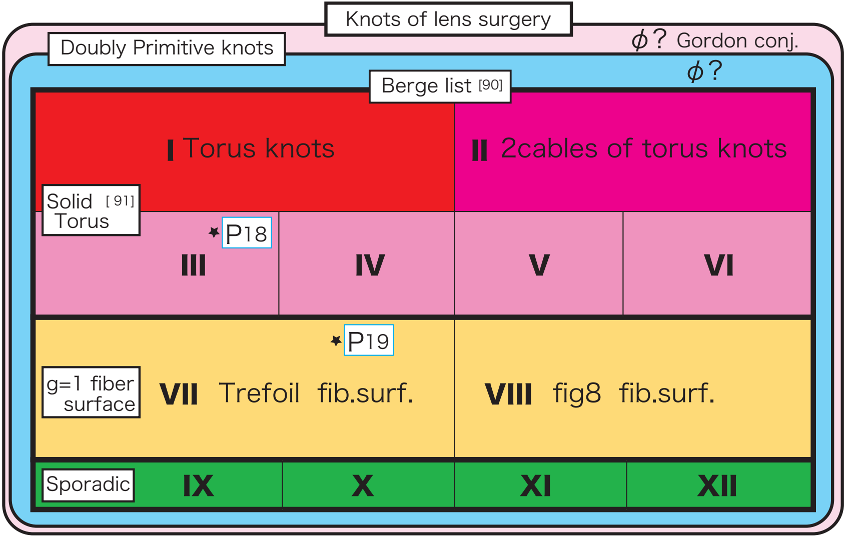
$K_\sharp$ (as in π_1) is a generator in both $\pi_1(H_2)$ and $\pi_1(-H_2)$.



Such a knot K with the surface slope (coeff.) always yields a lens space. ■

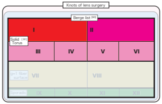
Berge (tried to) classfied and made a list of such knots.
His list consists of **3** Subfamilies (and of **12** "Type"s).

Type I, II, III, $\cdots$, VI | VII, VIII | IX, $\cdots$, XII.


Knots of lens surgery
Doubly Primitive knots
ϕ? Gordon conj.
ϕ?
Berge list [90]
I Torus knots
II 2cables of torus knots
Solid Torus [91]
★P18
III
IV
V
VI
★P19
g=1 fiber surface
VII Trefoil fib.surf.
VIII fig8 fib.surf.
Sporadic
IX
X
XI
XII

Berge's list (Subfamilies)

(1) Knots in the solid torus (Berge-Gabai knots)

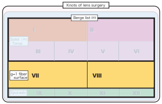


Type I : Torus knots

Type II : 2-cables of torus knots

Type III$\cdots$VI : "generic" lens surgery

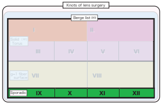
(2) Knots in the genus 1 fiber surface



Type VII : knots in the trefoil fiber surface F^+

Type VIII : knots in the fig8. fiber surface F^-

(3) Sporadic examples

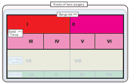


Type IX$\cdots$XII : Sporadic knots

(It is known XI = X, XII = IX)

§4. Results

Results : Divide presentation of Berge's knots



Theorem (1). ([Y ('09 AGT)])].
Every knot (up to mirror image)
in the subfamily (1), Type**I**, **II**, **III** $\cdots$ **VI**,

- is a *divide knot*,
- is presented by an *L-shaped plane curve* s.t.

$$\text{Area}(L) - \text{coeff.} \;=\; 0 \text{ or } 1. \qquad \square$$

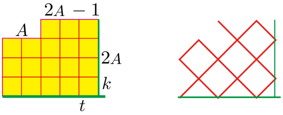
ex. Type**III** knots are parametrized by

$$\delta, \varepsilon \in \{\pm 1\},\; A(\geq 2), k(\geq 0) \text{ and } t \in \mathbf{Z}.$$

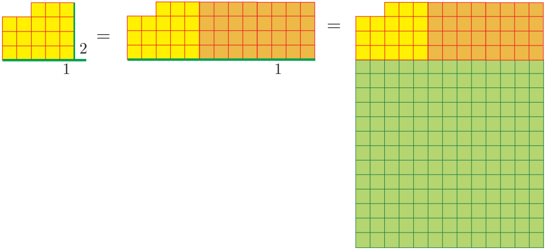
Thus we call the knots

$$k_{\mathbf{III}}(\delta, \varepsilon, A, k, t)$$

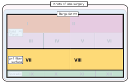
Type**III** knot: $k_{\mathbf{III}}(-1, +1, A, k, t)$
ex. $P(-2, 3, 7)$ (with 18-surgery) is $k_{\mathbf{III}}(-1, +1, 2, 0, 0)$,
ie, $A = 2, k = t = 0$. The divide presentation is



Green lines are used for $k, t > 0$, we add squares (full-twists).
Notation:

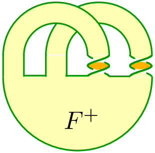


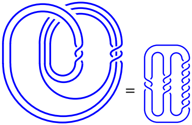
Theorem (2-1).([Y ('05 JKTR)])
TypeVII (L-shaped)
Let F^+ be the fiber surface of the left-handed trefoil.

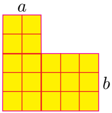


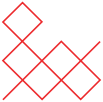
· [Y'10] Any TypeVII knot is $k^+(a, b)$ in F^+
with a positive coprime (a, b) s.t. $0 < a < b$.

Its p-surgery is $L(p, q)$.
$(p = a^2 + ab + b^2, q = -(a/b)^2 \bmod p)$



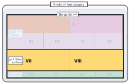






- $k^+(2, 3)$ is $P(-2, 3, 7)$. $2^2 + 2 \cdot 3 + 3^2 = 19$.

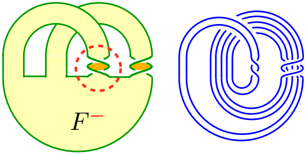
Theorem (2-2).
TypeVIII The most difficult.
Let F^- be the fiber surface of fig8 knot.



· [Y'10] Any TypeVIII knot is $k^-(a,b)$,
with a positive coprime (a,b) s.t. $0 < a < b/2$.

Its p-surgery is $L(p,q)$.
$(p = -a^2 + ab + b^2, q = -(a/b)^2 \bmod p)$

· It is known $k^-(a,b) = k^-(b-a,b)$.

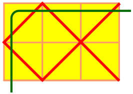


- $k^-(2,5)$ $\quad -2^2 + 2\cdot 5 + 5^2 = 31.$

On divide knots, negative twists are hard to treat with.

· $k^-(a, b)$ is a divide knot. The plane curve is constructed by a *blow-down* from the rectangle curve $a \times (b - a)$, as follows:

ex. $k^-(2, 5)$ is from $T(2, 3)$

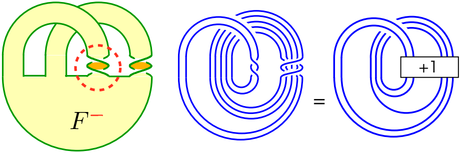


$\Rightarrow$ The curve

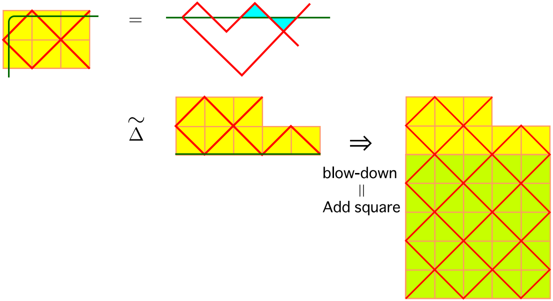
blow-down

Let's do the blow-down. □

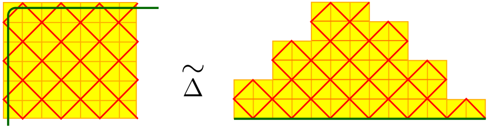
Key of the proof was ['05 Baker]'s deformation:



Blow-down from $T(2,3)$ to construct $k^-(2,5)$.

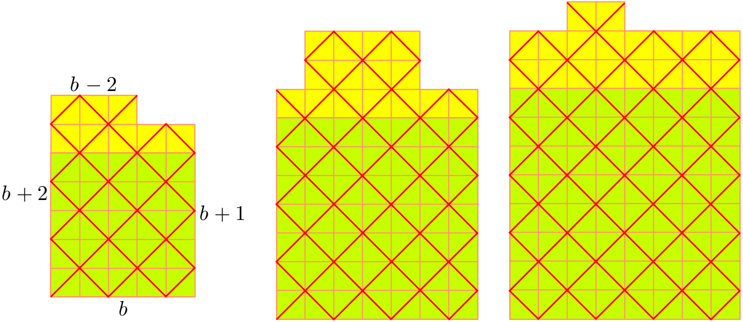


If a, b are large, it is a hard homework.



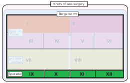
$\Rightarrow \ k^-(6,13)$.

TypeVIII knots.



$k^-(2,b)$ $\quad$ $k^-(3,7)$ $\quad$ $k^-(3,8)$

	$k^-(2,b)$	$k^-(3,7)$	$k^-(3,8)$
Area	b^2+2b-2	64	82
Coeff.	b^2+2b-4	61	79



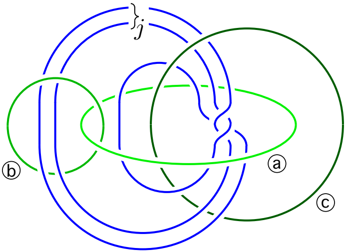
Theorem (3). Type IX and **X**

· (['05 Baker])

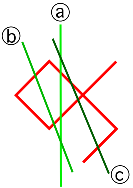
The knot $k_{\mathbf{IX}}(j)$ and $k_{\mathbf{X}}(j)$, $(j \in \mathbf{Z})$

is obtained from $T(j, j+1)$ by full-twists three times

for Type**IX**, in order (a $\to$ b $\to$ c), $p = 22j^2 + 9j + 1$,
for Type**X** , in order (a $\to$ c $\to$ b), $p = 22j^2 + 13j + 2$.

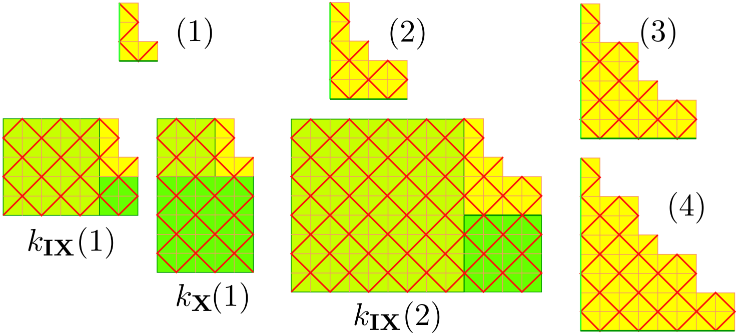


divide pres.



(j=2 case)

Sporadic knots in Type IX. $(p = 22j^2 + 9j + 1)$ and **Type X.**



$k_{\mathbf{IX}}(1)$, L-shaped, $p = 32$, Area=33.
$k_{\mathbf{IX}}(2)$, generalized L-shaped, $p = 107$, Area=109.

Fact. (maybe known to experts) $k^{-}(2,5)$ (in Type**VIII**, $p = 31$) is the same knot $k_{\mathbf{IX}}(1)$ ($p = 32$), but the coefficients p differ.

My work is not completed yet in the following sense:

Q1. Decide the final, best divide presentation, for every knots (Type**VIII**) in Berge's list of lens space surgery .
— minimal area, good-shaped, or...? —
The presentation is not unique, because of at least Δ-moves.

Related question:
Q2. For a given plane curve, determine whether it presents a knot of lens space surgery or not, and if it is, get the coefficient p (and Berge's parameter) from the curve.
Q3. Find more exceptional Dehn surgery along divide knots.

· ['09 Couture] introduced Khovanov "Categorification" invariants for divide knots...

Thank you very much!