4-manifolds constructed by lens space surgeries

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Intro

1.1. Thanks to

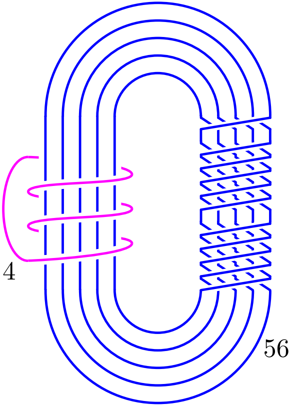
· [Saito-Teragaito 2010] (2008 Arxiv)
"Knots yielding homeomorphic lens spaces by Dehn surgery"

· 2010 Nov. "4-dim. topology" in Hiroshima University.
Sasahira: Instanton Floer homology over lens spaces

$\rightarrow$ Some $L(p,q)$ can not be smoothly embedded in $\mathbb{C}P^2\sharp\mathbb{C}P^2$.

[Sasahira] (2010)

1.2. Framed link



$$\begin{pmatrix} 56 & 15 \\ 15 & 4 \end{pmatrix}$$

$$T((5,11),(1,3)) = T(5,11) \cup T(3,1)$$

Framed link $(L;\mathbf{n}) \;=\; (K_1 \cup K_2 \cup \ldots \cup K_r \;;\; n_1, n_2, \ldots, n_r)$

Let K be a knot (L be a link) in $S^3 = \partial B^4$,
$n \in \mathbb{Z}$ (or $n = p/q \in \mathbb{Q} \cup \{1/0\}$)

Definition (3-manifold; Dehn surgery)

$$M^3(K;n) = (S^3 \backslash \mathrm{int} N(K)) \cup_{\varphi_n} (D^2 \times S^1)$$

where $\varphi_n : \partial(D^2 \times S^1) \to \partial\big(S^3 \backslash \mathrm{int} N(K)\big)$ is a homeomorphism s.t.

$$\varphi_{n*}(\partial D^2 \times \{\mathrm{pt}\}) = n[m] + [l] \quad (\mathrm{or}\ p[m] + q[l])$$

$[m], [l] \in H_1(S^3 \backslash \mathrm{int} N(K))$ is the meridian-longitude system.

Definition

The core $\{o\} \times S^1$ in $M(K;n)$ is called a dual knot of the surgery.

$n \in \mathbb{Z}$ (an integer) called *framing*.

Definition (4-manifold; Kirby diagram, 2-handle attach)

$$X^4(K;n) = B^4 \cup_{\overline{\varphi_n}} (D^2 \times D^2)$$

where $\overline{\varphi_n} : (\partial D^2) \times D^2 \hookrightarrow \partial B^4$ is an embedding s.t.

$$\overline{\varphi_n}(\partial D^2 \times \{o\}) = K,$$

$$\overline{\varphi_n}_*(\partial D^2 \times \{\mathrm{pt}\}) = n[m] + [l].$$

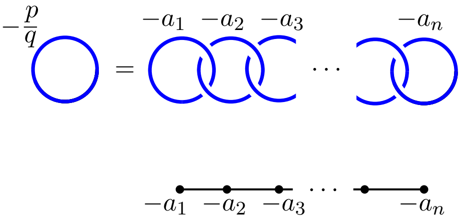
Fact

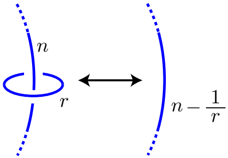
For $n \in \mathbb{Z}$,

$$\partial X^4(K;n) = M^3(K;n)$$

3-dim. Example: Lens space $L(p, q)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$

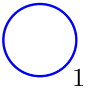




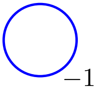
For $n \in \mathbb{Z}, r \in \mathbb{Q}$

$L(p, q) \cong L(p, q') \;\Leftrightarrow\; q' \equiv q$ or $q'q \equiv 1 \mod p$.

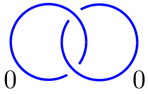
4-dim. Example: In this talk, we need only 2-handles.
We do not draw 4-handles



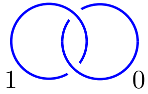
$\mathbb{C}P^2$



$\overline{\mathbb{C}P^2}$



$S^2 \times S^2$



$S^2 \tilde{\times} S^2$

A split sum describes a connected sum.

Dehn-Rolfsen move and Kirby calculus

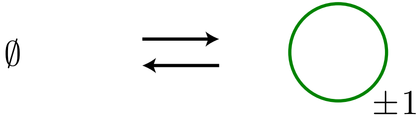
Theorem (Dehn-Rolfsen moves, Kirby calculus)

The 3-manifolds are homeo. $M(L;\mathbf{n}) \cong M(L';\mathbf{n}')$
$\Leftrightarrow$ *framed links* $(L;\mathbf{n}),(L';\mathbf{n}')$ *are moved to each other by*

(K1) Blow-up/down *(K2) Handle slide*

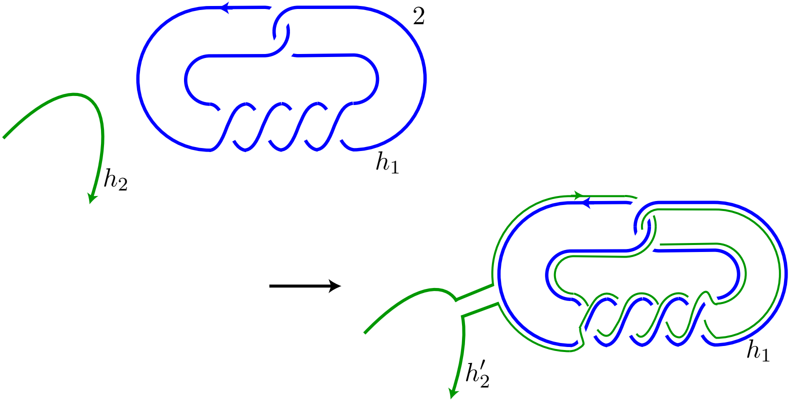
and isotopy.

(K1) Blow-up/ down



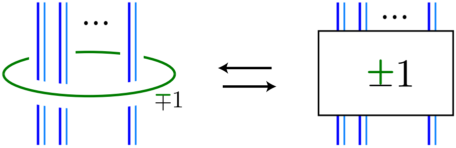
is related to add/remove $\overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2$

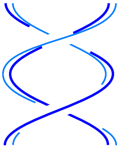
(K2) Handle slide



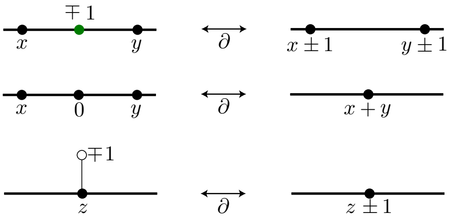
$n_2' = n_2 \pm 2\mathrm{lk}(h_1, h_2) + n_1$

Blow up in original sense (K1) + (K2)

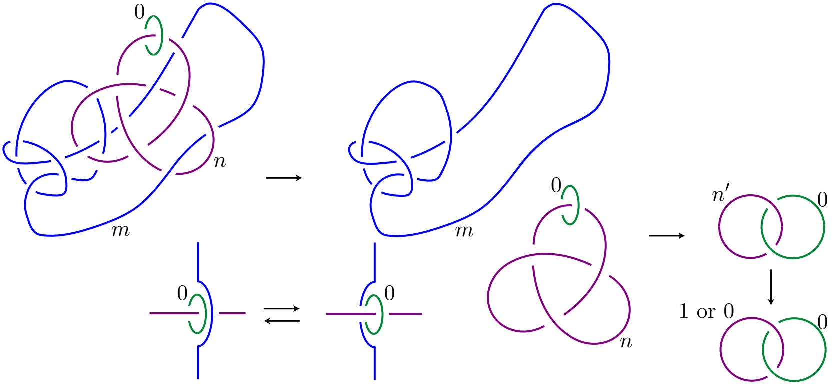




ex. +1 twist



(3-dim.) "A meridian 0" cancels the surgery on the component.
(4-dim.) "A meridian 0" means a $\sharp$ summand $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.



1.3. Lens space surgery

Q. Which knots K yield a lens space by Dehn surgery?

$$M(K; n) = L(P, Q)?$$

Note that $n = \pm P$.

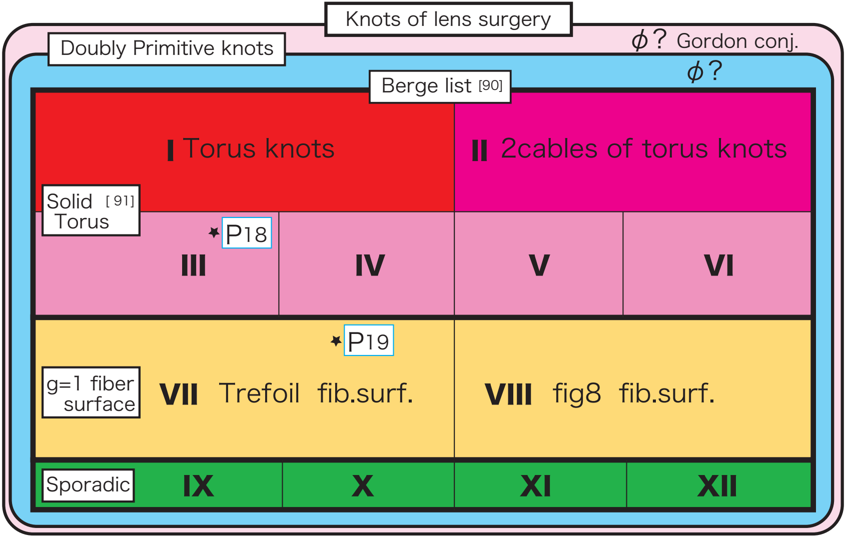
1. Torus knots

Theorem ([Moser '71])

$$M(T(p, q); pq \pm 1) = L(pq \pm 1, -p^2)$$

for each sign.

2. 2 cable of torus knots
3. Hyperbolic knots $\rightarrow$ *A Big Resarch!*


Knots of lens surgery
Doubly Primitive knots
ϕ? Gordon conj.
ϕ?
Berge list [90]
I Torus knots
II 2cables of torus knots
Solid [91]
Torus
P18
III
IV
V
VI
P19
g=1 fiber
surface
VII Trefoil fib.surf.
VIII fig8 fib.surf.
Sporadic
IX
X
XI
XII

[Saito-Teragaito] There exist some (a sequence of) pairs of knots

$$K \neq K' \text{ and } M(K;p) = L(p,q) = M(K';p),$$

for all pairs of types (ex. torus-hyperbolic), except cable-cable, of knots K, K'.

They also studied some examples

$$K \neq K' \text{ and } M(K;p) = L(p,q) = -M(K';p),$$

i.e., yielding orientation-reversed lens spaces.

1.3. Main Results

Our purpose is,

1. Study the 4-manifold

$$X^4(K;p) \cup \mp X^4(K';p).$$

In the case $M(K;p) = -M(K';p)$, the 4-manifold is 1-conn. and (the intersection form is) *definite*, thus is homeo. to $\pm(\mathbb{C}P^2 \sharp \mathbb{C}P^2)$, by Freedman's theorem.

2. Complete list of pairs (K, K') of torus knots s.t.

$$K \neq K' \text{ and } M(K;p) = L(p,q) = \pm M(K';p),$$

TABLE (Digest) $((S,T),(U,V))$ is in [Saito-Teragaito]

· Orientation-preserving (Bi-indexed by (a,i))

$$((A,B),(C,D)) \text{ with } P = AB+1 = CD-1$$

· Orientation-reversing I (Indexed by i)

$$((S,T),(U,V)) \text{ with } P = ST+1 = UV-1$$

$$((s,t),(u,v)) \text{ with } p = st+1 = uv-1$$

· Orientation-reversing II (Indexed by i)

$$((K,L),(M,N)) \text{ with } P = KL-1 = MN-1$$

$$((k,l),(m,n)) \text{ with } p = kl-1 = mn-1$$

Theorem (4-dim. problem)

For every pair of lens space surgeries along different torus knots yielding homeomorphic lens spaces,

$$K \neq K' \text{ and } M(K;p) = L(p,q) = \pm M(K';p),$$

the 4-manifold $X(K;p) \cup \mp X(K';p)$ *is diffeomorphic to the standard 4-manifold*

$$S^2 \times S^2, \quad S^2 \tilde{\times} S^2 \text{ or } \pm(\mathbb{C}P^2 \sharp \mathbb{C}P^2).$$

Corollary

Some Lens spaces can be smoothly embedded in $\mathbb{C}P^2 \sharp \mathbb{C}P^2$.

$$\begin{matrix} L(S_iT_i+1, S_i^2),\ L(K_iL_i-1, K_i^2) \\ L(s_it_i+1, s_i^2),\ L(k_il_i-1, k_i^2) \end{matrix} \hookrightarrow \mathbb{C}P^2 \sharp \mathbb{C}P^2.$$

[Sasahira] shows

$$\begin{array}{lcl} L(28657, 2) & \not\hookrightarrow & \mathbb{C}P^2 \sharp \mathbb{C}P^2, \\ L(28657, 7921) & \hookrightarrow & \mathbb{C}P^2 \sharp \mathbb{C}P^2. \end{array}$$

"If p is prime and $p \equiv 1 \mod 16$, ..."
$L(28657, 7921) = M(T(199, 144), 28657) = -M(T(89, 322), 28657)$.

Theorem (The table is complete)

Suppose that

$$M(K;p) = L(p,q) = \pm M(K';p')$$

is a pair of integral lens space surgeries along torus knots K, K'.

We retake K *as* $K = T(a,b)$ *with* $a, b > 0$, *by taking* $(K!; -p)$ *instead of* $(K;p)$, *and* K' *also.*

Then the retaken pair $((K;p),(K';p))$ *is in TABLE.*

TABLE

TABLE

$\boxed{((A,B),(C,D))}$: Let a be an integer $a \geq 3$.

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} C_0 \\ D_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A_{i+1} \\ B_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a-2 & 1 \end{bmatrix} \begin{bmatrix} C_i \\ D_i \end{bmatrix}, \quad \begin{bmatrix} C_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & a+2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}.$$

$[a=3]$

i	A	B	P	C	D
1	1	2	3	4	1
2	4	5	21	11	2
3	11	13	144	29	5
4	29	34	378	76	13
5	76	89	3117	199	34
⋮			⋮		

$[a=4]$

i	A	B	P	C	D
1	1	3	4	5	1
2	5	11	56	19	3
3	19	41	780	71	11
4	71	153	10864	265	41
⋮			⋮		

$$P = AB + 1 = CD - 1, \quad B^2 \equiv D^2 \mod P$$

$\boxed{((S, T), (U, V))}$ [Saito-Teragaito]

$$\begin{bmatrix} S_0 \\ T_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} S_{i+1} \\ T_{i+1} \end{bmatrix} = \begin{bmatrix} T_i \\ S_i + V_i \end{bmatrix}, \qquad \begin{bmatrix} U_{i+1} \\ V_{i+1} \end{bmatrix} = \begin{bmatrix} V_i \\ T_i + U_i \end{bmatrix}.$$

i	S	T	P	U	V
0	0	1	1	2	1
1	1	1	2	1	3
2	1	4	5	3	2
3	4	3	13	2	7
4	3	11	34	7	5
5	11	8	89	5	18
$\vdots$			$\vdots$		

$$P = ST + 1 = UV - 1, \qquad T^2 + U^2 \equiv 0 \mod P$$

$((s,t),(u,v))$

$$\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{pmatrix} s_{i+1} \\ t_{i+1} \end{pmatrix} = \begin{pmatrix} t_i \\ s_i + 2v_i \end{pmatrix}, \qquad \begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} v_i \\ 2t_i + u_i \end{pmatrix}.$$

i	s	t	p	u	v
0	0	1	1	1	2
1	1	4	5	2	3
2	4	7	29	3	10
3	7	24	169	10	17
4	24	41	985	17	58
5	41	140	5741	58	99
6	140	239	33461	99	338
$\vdots$			$\vdots$		

$$p = st + 1 = uv - 1, \quad t^2 + u^2 \equiv 0 \mod P$$

$((K,L),(M,N))$ Using a sequence $\{b_i\}$ defined by

$$b_1 = 1,\ b_2 = 2, \quad b_{i+1} = 3b_i - b_{i-1}\ (i \geq 2),$$

we define a pair $((K_i,L_i),(M_i,N_i))$ by

$$((K_i,L_i),(M_i,N_i)) \ = \ ((b_i, 3b_{i+1}),(b_{i+1}, 3b_i)).$$

i	K	L	P	M	N
1	1	6	5	2	3
2	2	15	29	5	6
3	5	39	194	13	15
4	13	102	1325	34	39
5	34	267	9077	89	102
$\vdots$			$\vdots$		

$$P = KL - 1 = MN - 1, \quad K^2 + M^2 \equiv 0 \mod P$$

$((k,l),(m,n))$ Using a sequence $\{d_i\}$ defined by

$$d_1 = 1,\ d_2 = 3, \quad d_{i+1} = 4d_i - d_{i-1}\ (i \geq 2),$$

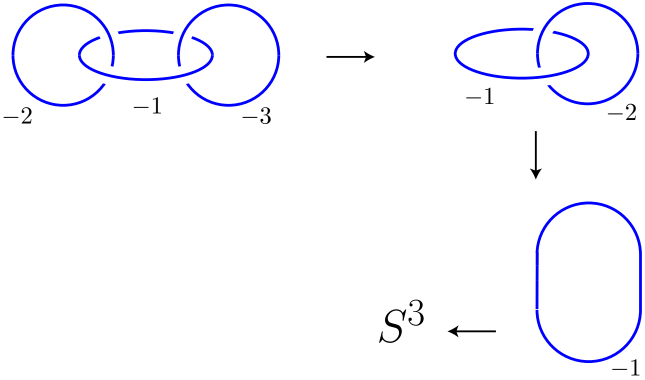
we define a pair $((k_i, l_i),(m_i, n_i))$ by

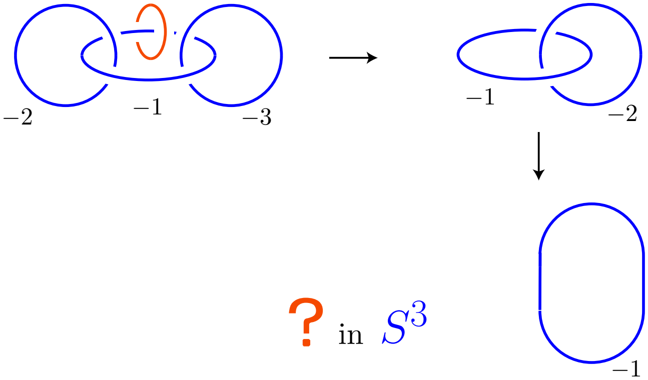
$$((k_i, l_i),(m_i, n_i)) \ = \ ((d_i, 2d_{i+1}),(d_{i+1}, 2d_i)).$$

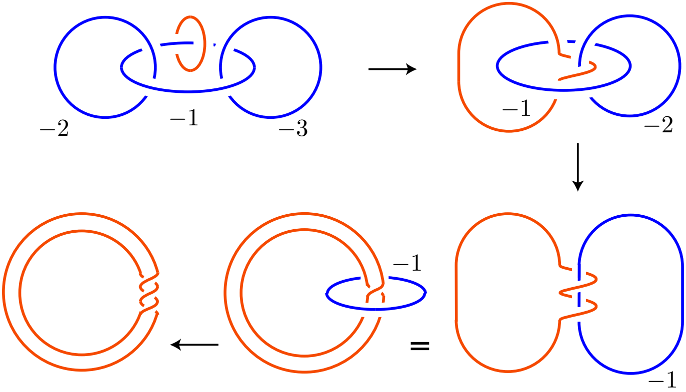
i	k	l	p	m	n
1	1	6	5	3	2
2	3	22	65	11	6
3	11	82	901	41	22
4	41	36	12545	153	82
5	153	1142	174725	571	306
$\vdots$			$\vdots$		

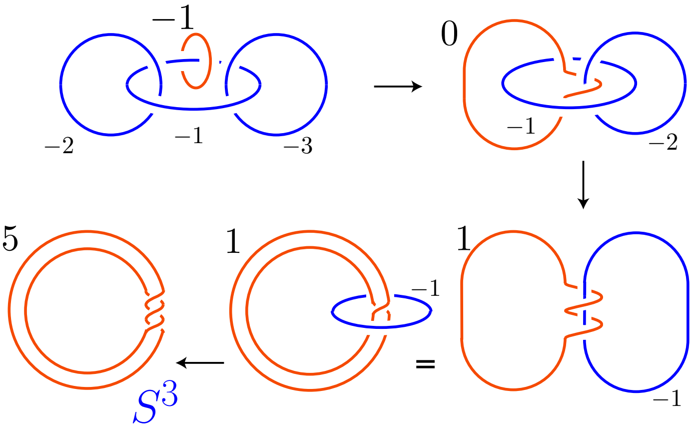
$$p = kl - 1 = mn - 1, \quad k^2 + m^2 \equiv 0 \mod P$$

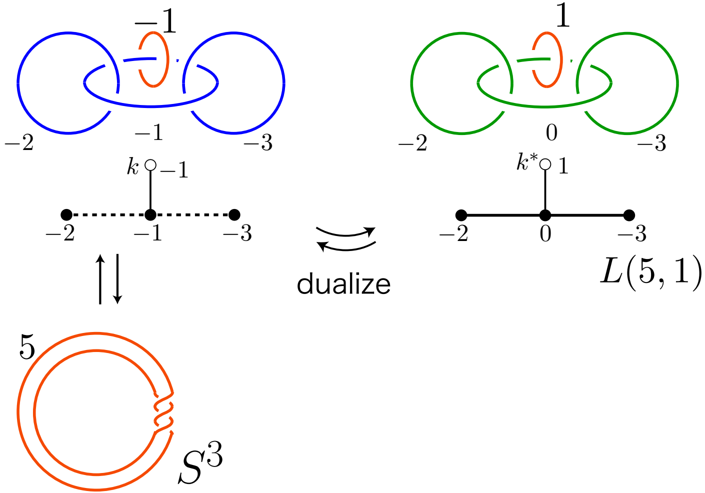
Lens space surgeries along Torus knots




−2
−1
−3
−1
−2
−1
? in S^3

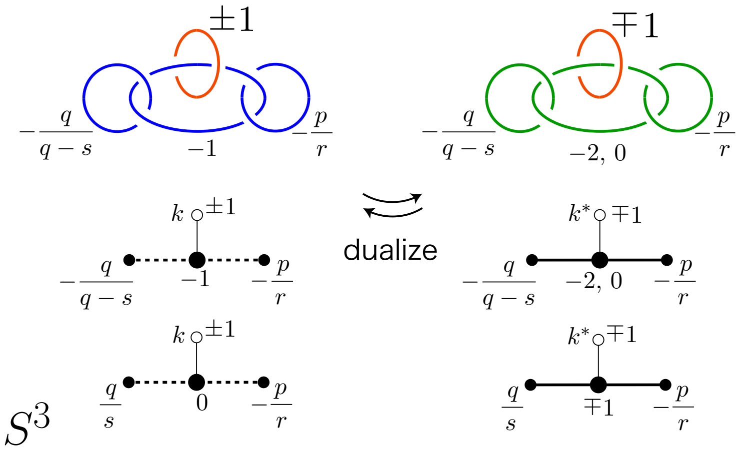

−2
−1
−3
−1
−2
−1
=
−1


−1
−2
−1
−3
0
−1
−2
5
1
−1
=
1
−1
S^3


−1
−2
−1
−3
k −1
−2
−1
−3
dualize
1
−2
0
−3
k* 1
−2
0
−3
L(5, 1)
5
S3

For a general $T(p,q)$, we use

$$ps - qr = 1, \qquad 0 < r < p,\ 0 < s < q$$



Outline of Proof

First of all, I want to say

Each family has its own personality

$((A, B), (C, D))$

$((S, T), (U, V))$

$((K, L), (M, N))$

4.1. Table gives the pairs

Lemma

The pairs (A_i, B_i) *and* (C_i, D_i) *satisfy the followings:*

(1) *Each pair* (A_i, B_i) *and* (C_i, D_i) *is coprime.*

(2) $A_iB_i + 1 = C_iD_i - 1$. *We call this integer* P_i.

(3) ${B_i}^2 \equiv {D_i}^2 \mod P_i$.

(4) *If* $a \geq 3$ *and* $i \geq 2$, *then* ${B_i}^4 \not\equiv \pm 1 \mod P_i$.

TABLE $((A, B), (C, D))$ and Definition (again).

$[a = 3]$

i	A	B	P	C	D
1	1	2	3	4	1
2	4	5	21	11	2
3	11	13	144	29	5
4	29	34	378	76	13
5	76	89	3117	199	34
⋮			⋮		

$[a = 4]$

i	A	B	P	C	D
1	1	3	4	5	1
2	5	11	56	19	3
3	19	41	780	71	11
4	71	153	10864	265	41
⋮			⋮		

$$\begin{bmatrix} A_{i+1} \\ B_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a-2 & 1 \end{bmatrix} \begin{bmatrix} C_i \\ D_i \end{bmatrix}, \qquad \begin{bmatrix} C_{i+1} \\ D_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & a+2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix}.$$

Lemma (*ABCD*'s Sublemma)

The pairs (A_i, B_i) *and* (C_i, D_i) *satisfy the followings:*

(1) $A_i + B_i - C_i + D_i = 0.$

(2) $(a-1)A_i - B_i - C_i + (a+1)D_i = 0.$

(3) *Each of* A_i, B_i, C_i *and* D_i *satisfies the same recursive formula:*

$$\mathcal{X}_{i+2} = a\mathcal{X}_{i+1} - \mathcal{X}_i, \qquad \text{for } \mathcal{X} = A, B, C, \text{or } D.$$

(5) *For any pair* $(\mathcal{X}, \mathcal{Y})$ *in* $\{A, B, C, D\}$, $\mathcal{X}_{i+1}\mathcal{Y}_i - \mathcal{Y}_{i+1}\mathcal{X}_i$ *is constant, i.e., does not depend on* i. *For example,* $A_{i+1}B_i - B_{i+1}A_i = a$ *and* $A_{i+1}D_i - D_{i+1}A_i = 2.$

(6) $B_i^2 - D_i^2 = (a-2)P_i$

4.2. Proof of 4-dim. Problem.

First 3-dim. calculus:
Where is the dual knot?
Where is the other dual knot?

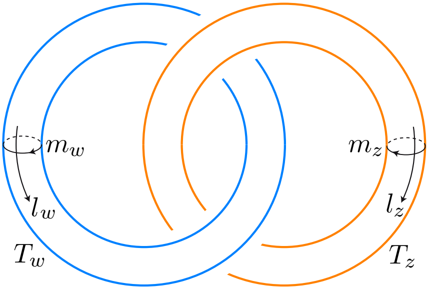
Next 4-dim. calculus: Only handle slides.

Definition (Cable of Hopf link)

We define a two-component link

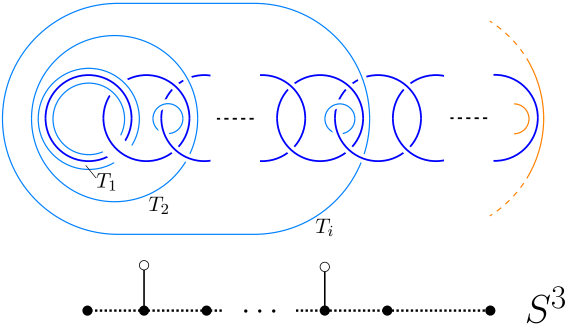
$$T((p_z, q_z), (p_w, q_w)) \ := \ T_z(p_z, q_z) \cup T_w(q_w, p_w),$$

$T_z(p_z, q_z)$ is a torus knot $p_z[l_z] + q_z[m_z]$ in T_z, and
$T_w(p_w, q_w)$ is a torus knot $q_w[l_w] + p_w[m_w]$ in T_w.
Their linking number is $p_z q_w$.



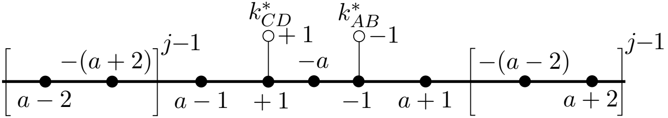
Lemma (parallel tori)

If a framed chain link below describes S^3, any pair $m_i \cup m_j$ of meridians is a link of type $T((p_z, q_z), (p_w, q_w))$.

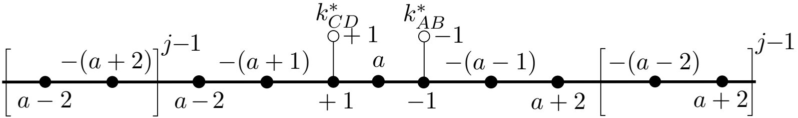


For $(\,(A(a,i), B(a,i)),\ (C(a,i), D(a,i))\,)$

Case $i = 2j$ is even,



Case $i = 2j+1$ is odd,



There is a symmetry.

Lemma (3-dim part : Pull back of the other dual knot)

The 4-manifold $X_{AB} \cup (-X_{CD})$ is described by the framed link

$$(T((A_i, B_i), (A_{i-1}, B_{i-1})); P_i, P_{i-1})$$

In other words, the 2-handle $(h^2_{CD})^{\perp}$ of $(-X_{CD})$ comes to the component $(T_w(B_{i-1}, A_{i-1}); P_{i-1})$.

Lemma (4-dim part : Kirby calculus)

There exists a sequence of handle-slides from

$$(T((A_i, B_i), (A_{i-1}, B_{i-1})); P_i, P_{i-1}) \text{ to }$$
$$(T((A_{i-1}, B_{i-1}), (A_{i-2}, B_{i-2})); P_{i-1}, P_{i-2}).$$

In other words, as 4-dim Kirby diagrams, they describe the same 4-manifold.

Proof. For a framed link in $T^2 \times [-1.5, 1.5]$

$$(K; r) = (l(p,q)[-1]; r), \quad (K_0; r_0) = (l(p_0, q_0)[0]; \varepsilon).$$

On the handle-slides of K over K_0, we have:

Lemma (Framed link in T^2)

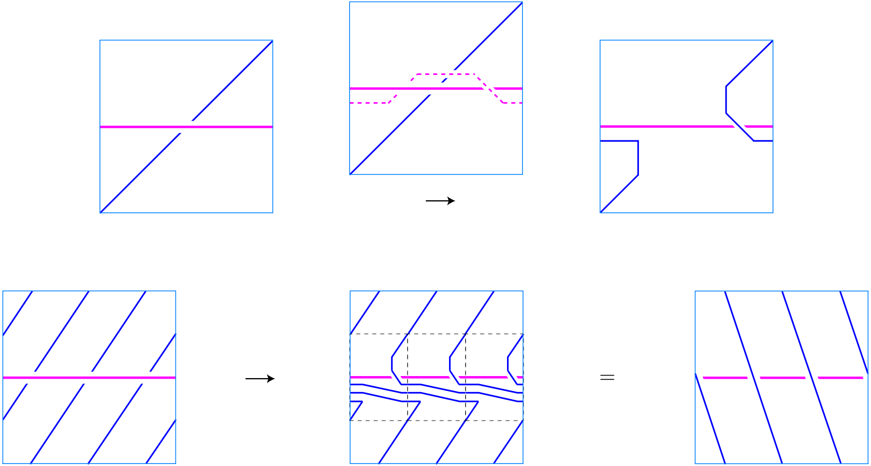
There exists a sequence of handle-slides of K over K_0 whose result is $(K_0; +1) \cup (K'; r')$ with

$$(K'; r') = (l(p - \varepsilon\Delta p_0, q - \varepsilon\Delta q_0)[+1]; r),$$

where $\Delta := p_0 q - q_0 p$.

Essentially, Picard-Lefshetz Theorem.
K_0 acts on K as a Dehn twist $D^{\varepsilon}_{K_0}$.

Handle slides in $T^2 \times I$



4.3. Proof of Completeness of the Table

$((S,T),(U,V)),((s,t),(u,v))$ case

Suppose that a given pair $((S,T),(U,V))$ satisfies

Condition L'

(0) (S,T) and (U,V) are coprime.

(1) $ST+1=UV-1$. We call this number P.

and also assume (by switching the entries if necessary) :

(2) $S^2+V^2 \equiv 0 \mod P$ (and $T^2+U^2 \equiv 0 \mod P$).

(3) $\min\{S,T,U,V\}=S$ or $\min\{S,T,U,V\}=U$. Possibly both (and $S=U$) hold.

By (2), define integers a, b by $S^2+V^2=aP$ and $T^2+U^2=bP$.

Definition (Reduction of a pair)

For a given pair $((S, T), (U, V))$ satisfying Condition L' (1), (2), (3) and $\min\{S, T, U, V\} > 2$, we define a pair $((S', T'), (U', V'))$ by

$$\begin{cases} S' = T - xU \\ T' = S \\ U' = V - xS \\ V' = U \end{cases} \quad \text{with } x = [T/U] = [V/S].$$

We call this operation from $((S, T), (U, V))$ to $((S', T'), (U', V'))$ *reduction*.

Note that (S', T'), (U', V') are possibly not coprime.

We can estimate $c = x \; = (TV - SU)/P.$ $ab = c^2 + 4.$

Lemma (Reduction keeps data until finish)

$((S', T'), (U', V'))$ *also satisfies Condition L' (1), (2), (3) and*

$$\{a', b'\} = \{b, a\}, \quad x' = x.$$

Either

(I) *It is the final reduction with* $\min\{S', T', U', V'\} \leq 2$ *below,*

$$(1,1,1,3), (1,1,3,1), (1,4,2,3), (1,4,3,2), (4,3,2,7), (2,2,6,1), (2,14,6,5), (s,s,1,s^2+2), (s,4s,2,2s^2+1), \quad (s > 1).$$

(II) *It still satidfies* $\min\{S', T', U', V'\} > 2.$

Thank you very much!