Difficulty on divide knot presentation of Type 8 knots in Berge's lens space surgery

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Four Dimensional Topology

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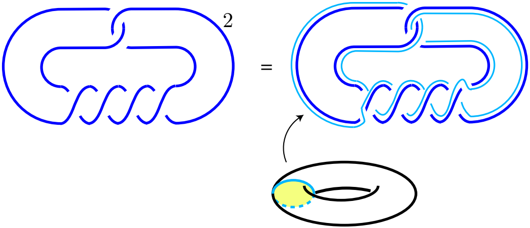
1. Introduction
2. Divide knots
3. Berge knots are divide knots
4. Results (Divide presentation of Type 8 knots)

§1. Dehn surgery

Dehn surgery = Cut and paste of a soliod torus.

$$(K;p) := \left(S^3 \setminus \text{open nbd} N(K)\right) \cup_\partial \text{Solid torus}.$$

Coefficient (in **Z**) "framing" = a *parallel* curve ($\subset \partial N(K)$) of K,
or the linking number.
Solid torus is reglued such as "the meridian comes to the parallel"



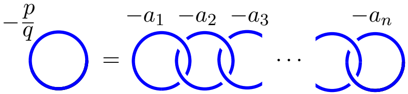
Theorem ([Lickorish '62])

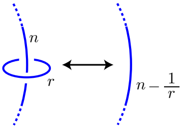
Any closed connected oriented 3-manifold M is obtained by a framed link $(L, \mathbf{p})$ in S^3, ie, $M = (L; \mathbf{p})$,

$$(L, \mathbf{p}) = (K_1, p_1) \cup (K_2, p_2) \cup \cdots \cup (K_n, p_n).$$

Lens space $L(p, q)$ $(p > q > 0)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$

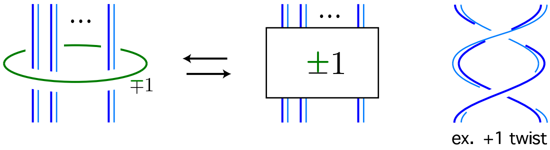




For $n \in \mathbf{Z}, r \in \mathbf{Q}$

Theorem (Kirby-Rolfsen moves (Fenn-Rourke's ver.))

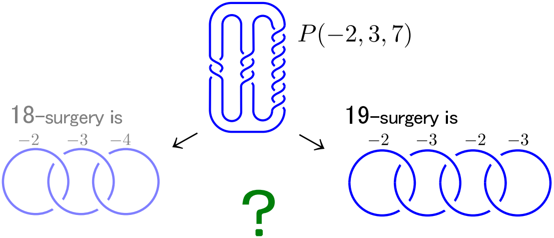
The 3-manifolds are homeo. $(L; \mathbf{p}) \cong (L'; \mathbf{p}')$
$\Leftrightarrow$ *framed links* $(L, \mathbf{p}), (L', \mathbf{p}')$ *are moved to each other by isotopy and the following*



Note: This (with a suitable sign) is blow-down /up, related to resolution of the singularity.

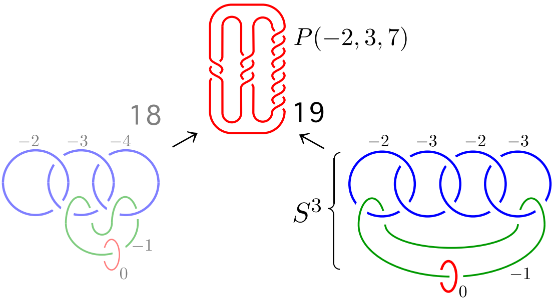
Kirby diagrams also present 4-manifolds (2-handle attachings).

Starting example of lens space surgery [Fintushel–Stern '80]
Which knot yields a lens space by Dehn surgery?



What is the best method to prove it?

My answer ([Y '05]) :



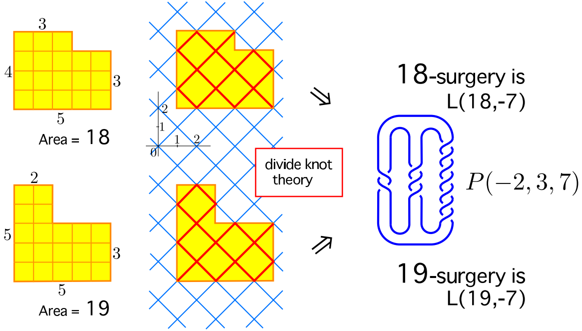
blue $\cup$ green $= S^3$, and red becomes the knot $P(-2,3,7)$.
The knot is constructed by seq. of full-twists = blow-downs.

In 2000, I heard A'Campo's divide theory (from Singularity theory)

a plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

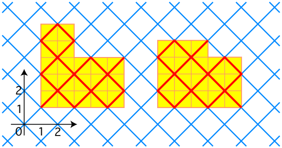
A'Campo's divide knots ['75]

Typical example of my results in [Y '05∼'20].

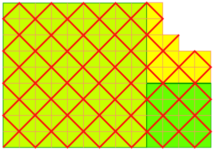


Examples. These examples presents knots of lens space surgery.

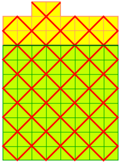
· L-shaped curve **I**$\sim$**VII**

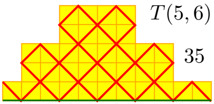


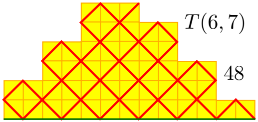
· (general.) L-shaped curve **IX**$\sim$**XII**



· T-shaped curve **VIII**





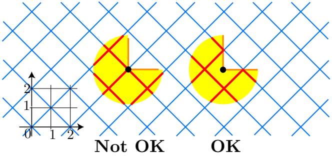


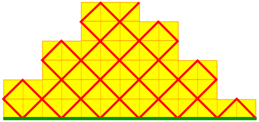
Here, green edge means *adding square(s)*.

We focus on curves cut from *the lattice*

Assume (on the regions, a union of rectangles)
- Vertices are in $\mathbf{Z}^2 \subset \mathbf{R}^2$,
- Edges of the region are *vertical* or *horizontal*,
- Every concave point is at odd point, and
- The curve P is an immersed arc ($\Leftrightarrow L(P)$ is a knot).

where we call a point $(m, n) \in \mathbf{Z}^2$ odd if $m + n \equiv 1 \mod 2$



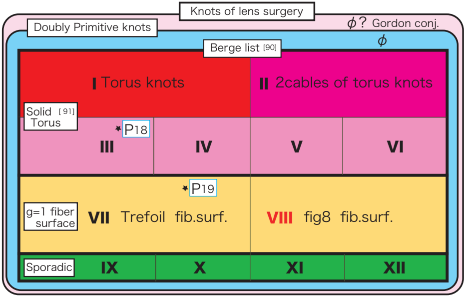


Berge's list ('90) Berge classfied and made a list of known lens space surgery.

Theorem (not published)

Every knot in Berge's list of lens space surgery is a divide knot.

Berge's list
Berge's list consists of
3 Subfamilies (1)(2)(3)
and of **12 Types**.



Type VIII is difficult!

By another approach (Heegarrd Floer homology, $\mathbb{C}$-links by Rudolph $\cdots$),

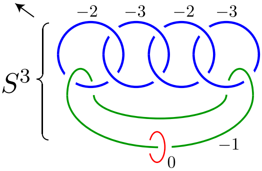
Theorem (Hedden '11)

Any knots of lens space surgery is intersection of an algebraic surface and B^4 in $\mathbb{C}^2$.

Theorem (Greene '13 : Lens space Realization Problem)

Lens spaces of lens space surgeries are classified.
Berge's list is complete, up to Heegaad Floer homology.

$(P;19)$



cf. Donaldson's diagonalization.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 3 \end{bmatrix}$$

§2. A'Campo's divide knots

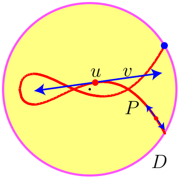
Original construction.

a generic plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

A'Campo's divide knots

Let P be a generic (no self-tangency) curve in the unit disk D,

$$S^3 = \{(u, v) \in TD | u \in D,\ v \in T_u D,\ |u|^2 + |v|^2 = 1\}$$
$$L(P) := \{(u, v) \in TD | u \in P,\ v \in T_u P,\ |u|^2 + |v|^2 = 1\} \subset S^3.$$



$y^2 = x(x^2 - \varepsilon)(x^2 - 2\varepsilon)$

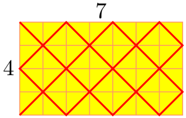
· A *strong-involution* $\iota : (u, v) \mapsto (u, -v)$.

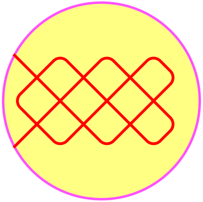
Ex. Torus links ['02 Goda-Hirasawa-Y, (Gusein-Zade, etc.)]

$P = B(p, q)$, the $p \times q$ rectangle billiard curve
(PL curve with slope ± 1 in the rectangle.)

$\Rightarrow$ $L(P)$ is $T(p, q)$

ex. $(p, q) = (7, 4)$





If $\gcd(p, q) = 1$, then $B(p, q)$ has $\frac{(p-1)(q-1)}{2}$ double points.

Basics on Divide knots [N.A'Campo, L.Rudolph,..]

(0) $L(P)$ is a knot ($\sharp L(P) = 1$) $\Leftrightarrow$ P is an immersed arc.

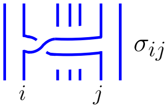
(1) The genus of knot $L(P)$ = $\sharp$ double points of P.

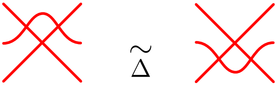
(2) For knots $L(P_1)$ and $L(P_2)$, $\mathrm{lk}(L(P_1), L(P_2)) = \sharp(P_1 \cap P_2)$.

(3) Every divide knot $L(P)$ is **fibered**.

(4) Any divide knot is a closure of *strongly quasi-***positive** braid. ie, product of some σ_{ij}.

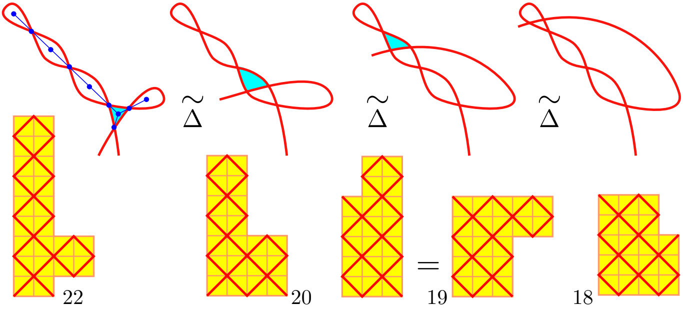
(5) $P_1 \sim P_2$ by Δ-move $\Rightarrow$ $L(P_1) = L(P_2)$.





Δ-move on plane curves

These curves present the same knot $P(-2,3,7)$
(Thanks to Hirasawa)



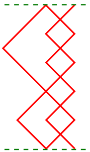
19 and 18 are the coefficients of lens space surgery.

['01 Couture-Perron] proved a viauallization.

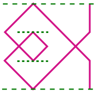
an OMD P $\Rightarrow$ a knot $L(P)$ in S^3

Braid presentation of $L(P)$

"Ordered Morse divide (OMD)" [Couture]



A divide is called *OMD*
if (w.r.t at least one direction) max/min points are in the same level, up to isotopy.



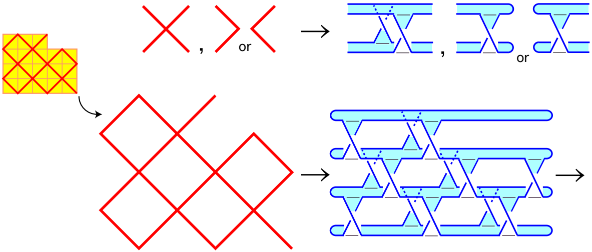
$\Leftarrow$ This is a Non-OMD.
(w.r.t horizontal nor vertical direction)

['01 Couture-Perron] proved a visualized version

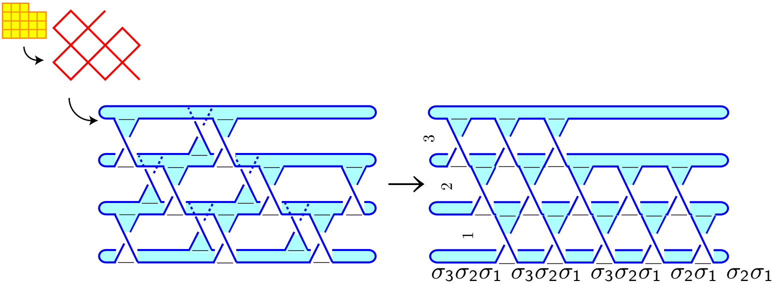
an OMD P $\Rightarrow$ a knot $L(P)$ in S^3

Braid presentation of $L(P)$

The braid presentation of $L(P)$.

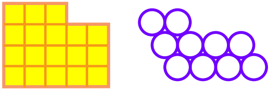


We get the fiber surface of $L(P)$. genus $= 5$.



The result is $W_4{}^3 W_3{}^2$, where

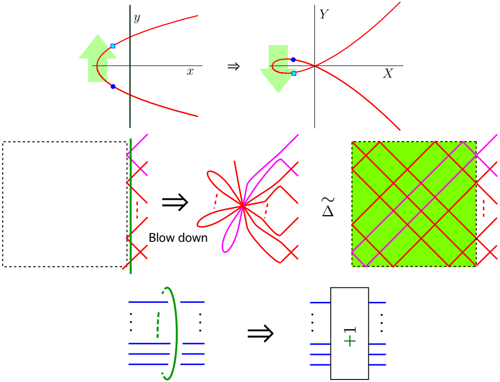
$$W_n := \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1 \quad \text{"}1/n\text{ twist"}.$$



Simple, only for L-shaped cases.

Lemma (Y)

"**Adding a square**" *corresponds to a right-handed* **full-twist** $=$ **blow-down** $=$ *coord. transform:* $(x, y) = (X, Y/X)$.
(ex. $y^2 = x + \epsilon$ *becomes* $Y^2 = X^2(X + \epsilon)$*)*

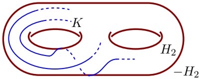


§3. Berge knots are divide knots

Which $(K;p)$ is a lens space?

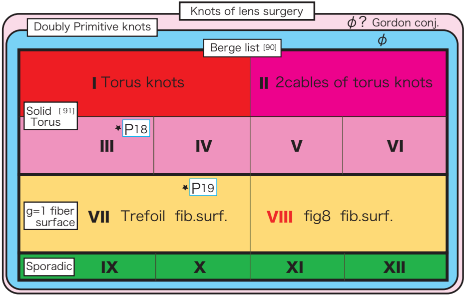
Berge's doubly-primitive knots ['90]
A knot K in the Heegaard surface Σ_2 is doubly-primitive iff

$K_\sharp$ (as in π_1) is a generator
in both $\pi_1(H_2)$ and $\pi_1(-H_2)$.



Such a knot K with the surface slope (coeff.) always yields a lens space.
■

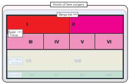
Berge's list
Berge classfied and made a list of such knots.
His list consists of
3 Subfamilies (1)(2)(3)
and of **12 Types**.



Theorem (Y '07∼'20)

Every knot in Berge's list, up to mirror image, is a divide knot.
Except Type **VIII**, *it is presented by an (a general.) L-shaped curve.*

Subfamily (1). [Y '09].
Every knot (up to mirror image)
in the subfamily (1), Type**I**, **II**, **III** $\cdots$ **VI**,
· is a divide knot, · is presented by an L-shaped curve.

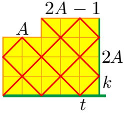


$$\mathrm{Area}(L) - \mathrm{coeff.} \;=\; 0 \text{ or } 1. \qquad \square$$

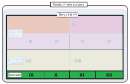
ex. Type**III** knots are parametrized by

$$\delta, \varepsilon \in \{\pm 1\},\ A(\geq 2),\quad k(\geq 0) \text{ and } t \in \mathbf{Z}.$$

$P(-2,3,7)$ (with 18-surgery) is $k_{\mathbf{III}}(-1,+1,2,0,0)$,
ie, $A = 2, k = t = 0$.



Subfamily (3). [Y '20] **Type IX** and **X**



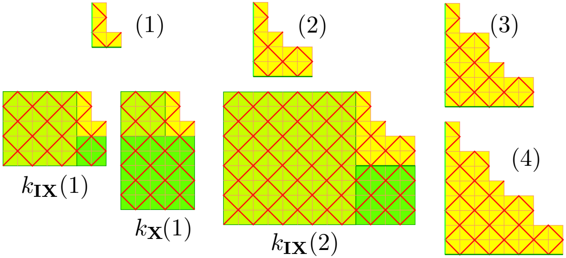
The knot $k_{\mathbf{IX}}(j)$ and $k_{\mathbf{X}}(j)$, $(j \in \mathbf{Z})$

· is a divide knot, · is presented by a general. L-shaped curve.

· is obtained from L-shaped $T(j, 2j+1)$ by full-twists twice.

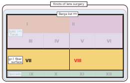
for Type**IX**, in order (bottom, left), $p = 22j^2 + 9j + 1$,

for Type**X** , in order (left, bottom), $p = 22j^2 + 13j + 2$.



Subfamily (2-1). [Y'05] **TypeVII**

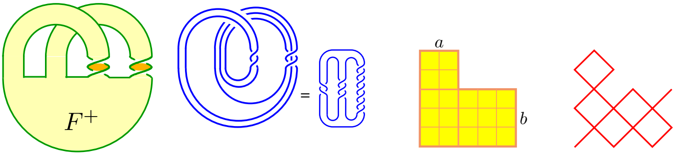
Let F^+ be the fiber surface of the left-handed trefoil.



· [Y'10] Any TypeVII knot is $k^+(a,b)$ in F^+ with a positive coprime (a,b) s.t. $0 < a < b$.

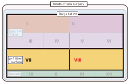
Its p-surgery is $L(p,q)$. ($p = a^2 + ab + b^2, q = -(a/b)^2 \mod p$)

· is a divide knot, · is presented by an L-shaped curve.



- $k^+(2,3)$ is $P(-2,3,7)$. $2^2 + 2 \cdot 3 + 3^2 = 19$.

Theorem (2-2). TypeVIII



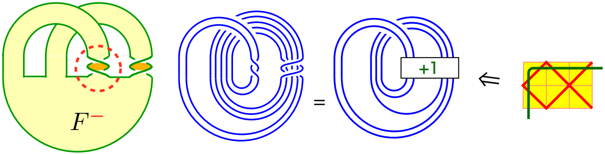
Let F^- be the fiber surface of fig8 knot.

· [Y'10] Any TypeVIII knot is $k^-(a,b)$,
with a positive coprime (a,b) s.t. $\underline{0 < a < b/2}$.

Its p-surgery is $L(p,q)$. ($p = -a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)

· is a divide knot **Q.** What type of curves?

· It is known $k^-(a,b) = k^-(b-a,b)$. ['05 Baker]'s deformation.

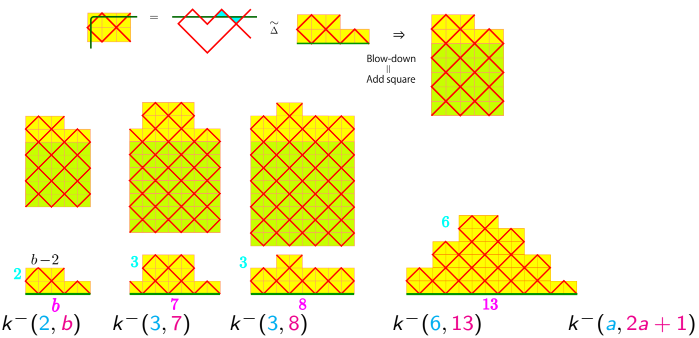


• $k^-(2,5)$ $-2^2 + 2 \cdot 5 + 5^2 = 31$.

On divide knots, negative twists are hard to treat with.

· $k^-(a, b)$ is a divide knot. The plane curve is constructed by a blow-down from $B(a, b-a)$ along *the broken curve*.

ex. Blow-down from $T(2,3)$ to $k^-(2,5)$. Couture's move

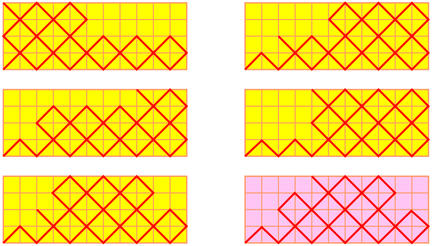


Conj. $k^-(a, b)$ is presented by a blow down along *the bottom edge* from a **T-shaped curve** of height a, width b and presenting $T(a, b-a)$.

Conj. $k^{-}(a, b)$ is presented by a blow down along the bottom edge from a **T-shaped curve** of height a, width b presenting $T(a, b-a)$.

Fact (Main Results [Y])

There exist only five **T-shaped curves** *of height 4, width 11 and presenting a knot with* $g = 9 \ = g(T(4,7))$*.*
No one is $T(4,7)$*, except the trivial one.* $\Rightarrow$ **Conj.** *is false.*



Final Remarks.

Question (1)

Decide the best divide presentation, for every knots (Type **VIII***) in Berge's list of lens space surgery.*

Question (2)

How about the other exceptional surgeries?

Question (3) (by J.Greene)

Restricted to divide knots, is Berge's conj. true?

Question (4)

Plane curves with cusp as divide [Sugawara-Yoshinaga]

謝辞

I would like to thank to

Lens space surgery

合田洋 先生、茂手木公彦 先生、寺垣内政一 先生、Ken Baker 氏、

Divide knots

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O. Couture 氏、

Thank you very much!