Dehn surgery: Topic around Type 7 and 8 knots

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微分トポロジー '22　デーン手術
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2. Lens space surgery
3. Related stories
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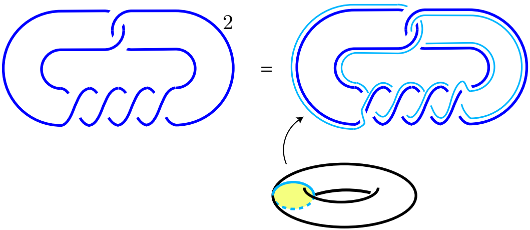
5. Recent Topic

§1. Dehn surgery

Dehn surgery $=$ Cut and paste of a soliod torus.

$$(K; p) := \left(S^3 \setminus \text{open nbd} N(K)\right) \cup_\partial \text{Solid torus.}$$

Coefficient (in $\mathbf{Z}$) "framing" $=$ a *parallel* curve ($\subset \partial N(K)$) of K,
or the linking number.
Solid torus is reglued such as "the meridian comes to the parallel"



Theorem ([Lickorish '62])

Any closed connected oriented 3-manifold M is obtained by a framed link $(L, \mathbf{p})$ in S^3, ie, $M = (L; \mathbf{p})$,

$$(L, \mathbf{p}) \;=\; (K_1, p_1) \cup (K_2, p_2) \cup \cdots \cup (K_n, p_n).$$

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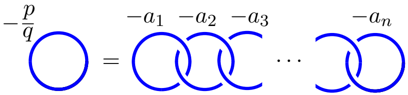
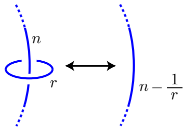
Lens space $L(p, q)$ $(p > q > 0)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$

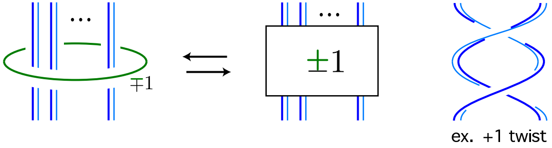




For $n \in \mathbf{Z}$, $r \in \mathbf{Q}$

Theorem (Kirby-Rolfsen moves (Fenn-Rourke's ver.))

The 3-manifolds are homeo. $(L;\mathbf{p}) \cong (L';\mathbf{p}')$
$\Leftrightarrow$ *framed links* $(L,\mathbf{p}),(L',\mathbf{p}')$ *are moved to each other by isotopy and the following*

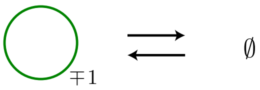


Note: This (with a suitable sign) is blow-down /up, related to resolution of the singularity.

Kirby diagrams also present 4-manifolds (2-handle attachings).

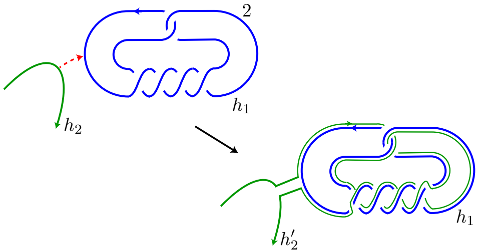
As 4-manifolds,

(K1) Blow-down /up



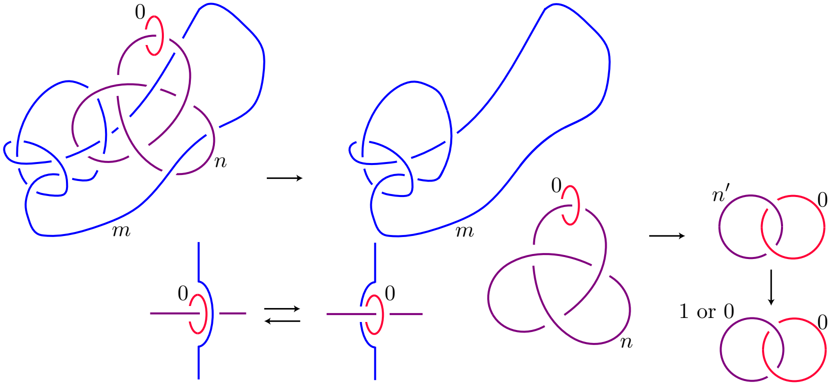
is related to remove /add $\overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2$

(K2) Handle slide

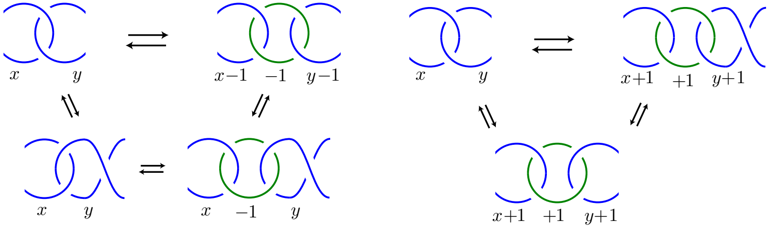


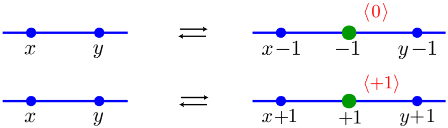
$$n_2' = n_2 \pm 2\mathrm{lk}(h_1, h_2) + n_1$$

(3-dim.) A meridian 0 cancels the surgery on the component.
(4-dim.) A meridian 0 means a ♯ summand $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.



For lens spaces, they are useful.





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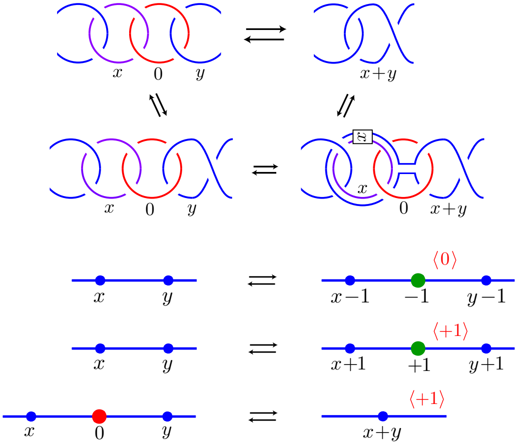

Let K be a knot (L be a link) in $S^3 = \partial B^4$ and $p \in \mathbb{Z}$

Definition (Dehn surgery (as before))

$$M(K,p) = (K;p) = \left(S^3 \backslash \mathrm{int} N(K)\right) \cup_{\varphi_p} (D^2 \times S^1)$$

where $\varphi_p : \partial(D^2 \times S^1) \to \partial\left(S^3 \backslash \mathrm{int} N(K)\right)$ is a homeomorphism s.t.

$$\varphi_{p*}(\partial D^2 \times \{\mathrm{pt}\}) = p[m] + [l] \qquad (\mathrm{or}\ p[m] + q[l])$$

$[m], [l] \in H_1(S^3 \backslash \mathrm{int} N(K))$ is the meridian-longitude system.

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Definition

The core $\{o\} \times S^1$ in $M(K,p)$ is called a dual knot of the surgery.

$$\varphi_{p*}(\partial D^2 \times \{\mathrm{pt}\}) = p[m] + [l]$$

$n \in \mathbb{Z}$ (an integer) called *framing*.

Definition (4-manifold; Kirby diagram, 2-handle attach)

$$X^4(K, p) = B^4 \cup_{\overline{\varphi_p}} (D^2 \times D^2)$$

where $\overline{\varphi_p} : (\partial D^2) \times D^2 \hookrightarrow \partial B^4$ is an embedding s.t.

$$\overline{\varphi_p}(\partial D^2 \times \{o\}) = K,$$

$$\overline{\varphi}_{p*}(\partial D^2 \times \{\mathrm{pt}\}) = p[m] + [l]$$

Fact

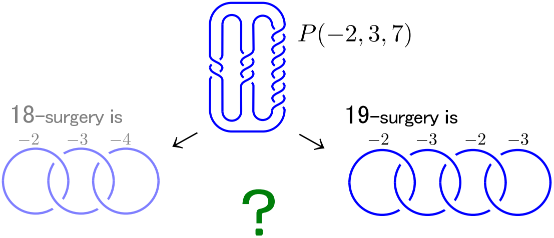
For $p \in \mathbb{Z}$,

$$\partial X^4(K, p) = M^3(K, p)$$

Note that $\partial(D^2 \times D^2) = (\partial D^2) \times D^2 \cup D^2 \times (\partial D^2)$.

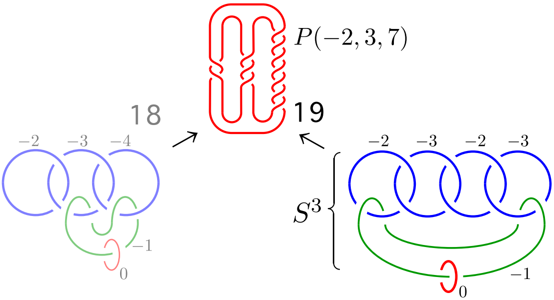
§2. Lens space surgery

Starting example of lens space surgery [Fintushel–Stern '80]
Which knot yields a lens space by Dehn surgery?



What is the best method to prove it?

My answer ([Y '05]) :



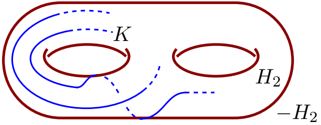
blue $\cup$ green $= S^3$, and red becomes the knot $P(-2,3,7)$.
The knot is constructed by seq. of full-twists $=$ blow-downs.

The green is the dual knot.

Berge's doubly-primitive knots ['90]

A knot K in the Heegaard surface Σ_2 is doubly-primitive iff

$$K_\sharp \text{ (as in } \pi_1) \text{ is a generator in both } \pi_1(H_2) \text{ and } \pi_1(-H_2).$$

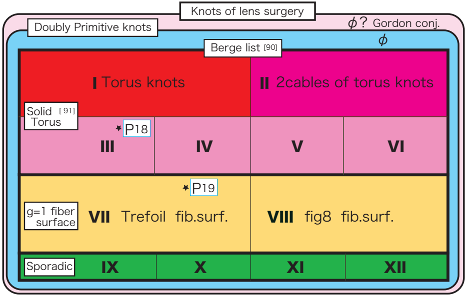


Such a knot K with the surface slope (coeff.) always yields a lens space. ■

Berge (tried to) classfied and made a list of such knots.
His list consists of **3** Subfamilies (and of **12** "Type"s).

Type I, II, III, $\cdots$, VI | VII, VIII | IX, $\cdots$, XII.

Berge's list ('90) J. Berge classfied and made a list of known lens space surgery. Berge's list consists of **3** Subfamilies (1)(2)(3) and of **12 Types**.



Theorem ([Moser '71])

$$(T(p,q); pq \pm 1) = -L(pq \pm 1, p^2)$$

$$p^2 q^2 \equiv 1 \mod pq \pm 1$$

Parameters ([Lisca '07, Greene '13])

寺垣内先生の講演
斎藤さんの k
The dual knot is $(1,1)$-bridge in $L(p,q)$

Type	**I** Torus knots **II** 2-cables are omitted
III	(a)$_\pm$ $p \equiv \pm(2k-1)d \mod k^2,\ d \mid k+1,\ (k+1)/d$ odd (b)$_\pm$ $p \equiv \pm(2k+1)d \mod k^2,\ d \mid k-1,\ (k-1)/d$ odd
IV	(a)$_\pm$ $p \equiv \pm(k-1)d \mod k^2,\ d \mid 2k+1$ (b)$_\pm$ $p \equiv \pm(k+1)d \mod k^2,\ d \mid 2k-1$
V	(a)$_\pm$ $p \equiv \pm(k+1)d \mod k^2,\ d \mid k+1,\ d$ odd (b)$_\pm$ $p \equiv \pm(k-1)d \mod k^2,\ d \mid k-1,\ d$ odd
VII	$k^2+k+1 \equiv 0 \mod p$
VIII	$k^2-k-1 \equiv 0 \mod p$
IX	$p = (2k^2+k+1)/11,\ k \equiv 2 \mod 11$
X	$p = (2k^2+k+1)/11,\ k \equiv 3 \mod 11$

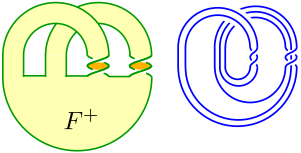
Type**VII** ($\varepsilon = +$) and **VIII** ($\varepsilon = -$) knots : $k^{\varepsilon}(a, b)$

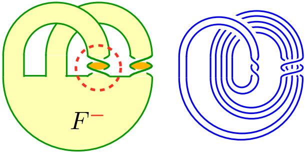
Let F^{+} be the fiber surface of the left-handed trefoil, F^{-} that of Fig8 knot.
The knot $k^{\varepsilon}(a, b)$ is defined as a (a, b)-curve in F^{ε}.

Lemma ([J.Berge '90])

$$(k^{\varepsilon}(a, b); P) = L(P, Q),$$

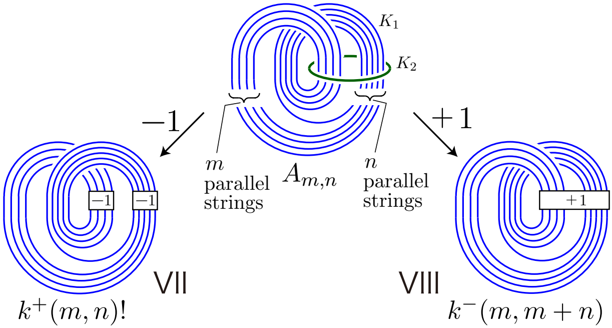
where $P = \varepsilon a^2 + ab + b^2$, $Q = -(a/b)^2$ *in* $(\mathbb{Z}/P\mathbb{Z})^*$.





- $k^{+}(2, 3)$ $P = 2^2 + 2 \cdot 3 + 3^2 = 19$
- $k^{-}(2, 5)$ $P = -2^2 + 2 \cdot 5 + 5^2 = 31.$

Remark. On the link $A_{m,n}$ in [門上-Y], -1 (or $+1$) full-twist of K_1 along K_2 is *directly* related to knots of lens space surgery **Type VII** (or **VIII**, [Baker]).



From $A_{m,n}$, we get

$k^+(m,n)$	m^2+mn+n^2 -surgey	**VII**
$k^-(m,m+n)$	$-m^2+mn+n^2$ -surgey	**VIII**

After Heegarrd Floer homology, $\mathbb{C}$-links by Rudolph $\cdots$,

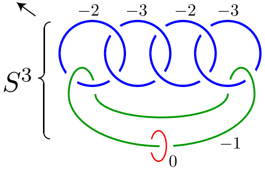
Theorem (Hedden '11)

Any knots of lens space surgery is intersection of an algebraic surface and B^4 in $\mathbb{C}^2$.

Theorem (Greene '13 : Lens space Realization Problem：丹下さんの講演)

Lens spaces of lens space surgeries are classified.
Berge's list is complete, up to Heegaad Floer homology.

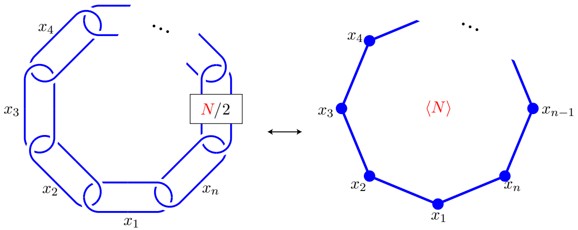
$(P; 19)$



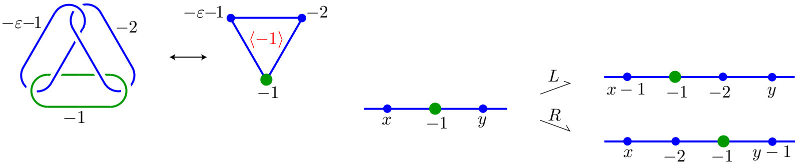
cf. Donaldson's diagonalization.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 3 \end{bmatrix}$$

Drawing notation:

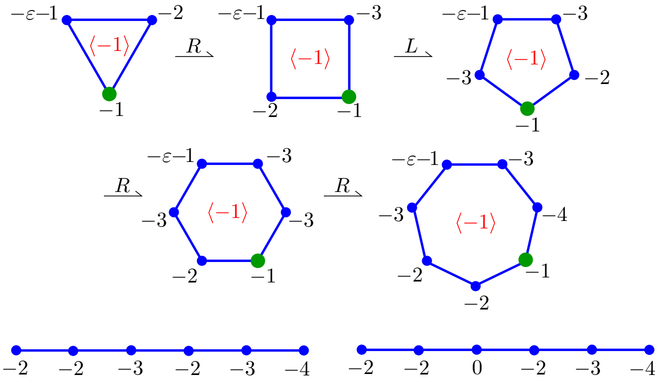


How to prove $(k^{\varepsilon}(a, b); P) = L(P, Q)$:



Euclidean Algorithm : For $(a, b) = (4, 7)$, we have $w(4, 7) = RLRR$ by

$$(4, 7) \to_R (4, 3) \to_L (1, 3) \to_R (1, 2) \to_R (1, 1)$$



□

§3. Related Stories

(1) **合田–寺垣内 Conj.** '00
For lens space surgeries along hyperbolic knots K

$$2g(K) + 8 \leqq p \leqq 4g(K) - 1$$

- ['07 P.B.Kronheimer-T.S.Mrowka-P.Ozsváth-Z.Szabó]

$$2g(K) - 1 \leqq p$$

- ['11 M.Hedden] considered rk$\widehat{HFK}(K)$ for "$<$"case and "$=$"case above, respectively.
- ['13 J.Greene]

$$2g(K) - 1 \leqq p - 2\sqrt{(4p+1)/5}$$

  the equality holds for $k^{-}(n, 2n+1)$ with $p = 5n^2 + 5n + 1$, in TypeVIII.

(2) The $K = P(-2, 3, 7) = k^+(2, 3)$'s Alexander Polynomial

$$\Delta(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1$$

$\Delta(-t)$ is known as "Lehmer's polynomial".

Fact (Y, 門上-Y)

$(K; 19) = -L(19, 7) = -L(19, 11)$

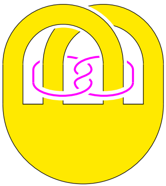
$$\Delta(t) \cdot \Delta(t^7) \cdot \Delta(t^{11}) = 1 \qquad in \ \ \mathbb{Z}[t]/ < t^{19} - 1 > .$$

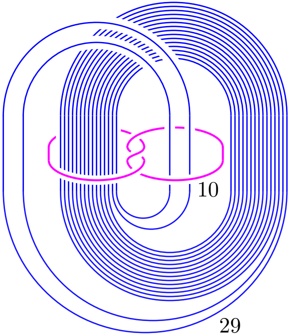
Q? *Let* $P = a^2 + ab + b^2$ *and* $\omega = a/b \in (\mathbb{Z}/P\mathbb{Z})^*$. *Then* $\omega^3 = 1$.
$(k^+(a, b); P) = -L(P, \omega) = -L(P, \omega^2)$

$$\Delta(t) \cdot \Delta(t^\omega) \cdot \Delta(t^{\omega^2}) \doteq 1 \qquad in \ \ \mathbb{Z}[t]/ < t^P - 1 > .$$

This is generalized in [門上-Y].

(3) **Framed link** describes 3-dim and 4-dim manifolds





$$\begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix}$$

ex. $C(2,15) = T(2,15) \cup T(2,3)$

Q. What is this manifold ?

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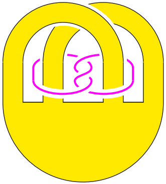
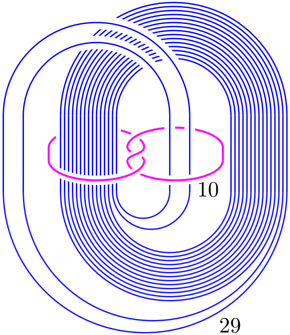




$$\begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix}$$

ex. $C(2,15) = T(2,15) \cup T(2,3)$

Answer. S^3 as 3-dim. $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ as 4-dim.
It also shows $L(29,4) \hookrightarrow \mathbb{C}P^2 \sharp \mathbb{C}P^2$.

Corollary

Some Lens spaces can be smoothly embedded in $\mathbb{C}P^2\sharp\mathbb{C}P^2$.

$$\textit{Some } L(P,Q)s \hookrightarrow \mathbb{C}P^2\sharp\mathbb{C}P^2.$$

[Sasahira] shows

$$\begin{array}{ll} L(28657,2) & \not\hookrightarrow \mathbb{C}P^2\sharp\mathbb{C}P^2, \\ L(28657,7921) & \hookrightarrow \mathbb{C}P^2\sharp\mathbb{C}P^2. \end{array}$$

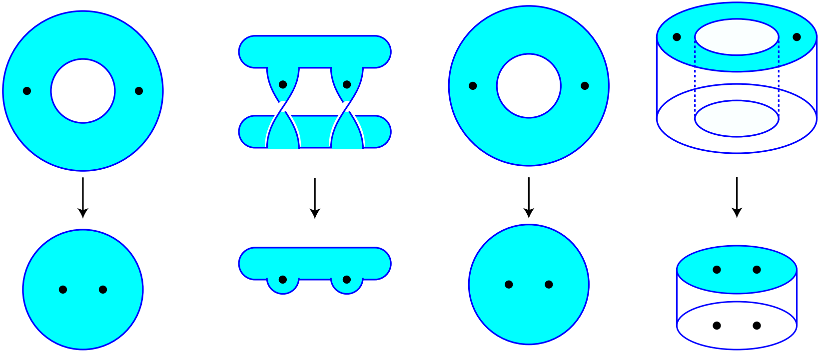
"If p is prime and $p \equiv 1 \mod 16$, ..."
$L(28657,7921) = (T(199,144);28657) = -(T(89,322);28657)$.

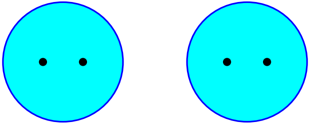
§4-1. (Add) Montesinos trick

Montesinos trick

If a knot K admits a strong-involution $\iota : (S^3, K) \to (S^3, K)$, we can use (the base space of) 2 branched covering $S^3/\iota \cong S^3$.

Dehn surgery $(K; p)$ from S^3 corresponds to local change of K/ι in S^3/ι.





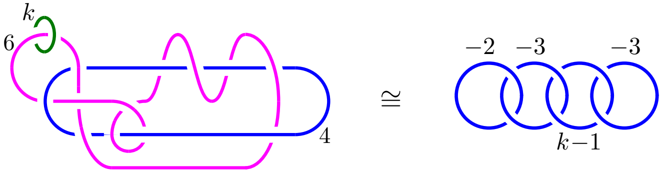
§4. Recent Topic : Baker-Luecke's work

As a previous research

Fact (Dunfield-Hoffman-Licata '15)

There exist infinitely many 1-cusped hyperbolic 3-manifolds which are **asymmetric** *and have two lens space fillings of coprime order. Specifically, for all large $k \in \mathbb{Z}$, the $(6k \pm 1, k)$ Dehn filling on either yields such a manifold.*

Dunfield-Hoffman-Licata link ['15]



Theorem (Baker-Luecke '20)

There exist asymetric L-space knots in S^3.

Conjecture by Lidman-Moore, and by Watson is solved negatively.

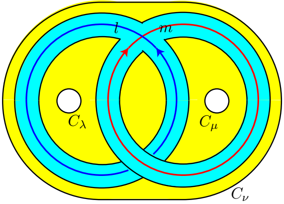
More precisely,

Fact (Baker-Luecke '20)

There exists surgery along a hyperbolic **knot** *satisfying*
(i) it is realized as a surgery along a strongly invertible **link** *such that the result is the double branched cover of an alternating link, and*
(ii) isometry group is trivial, thus the **knot** *is not strongly invertible.*

The construction is by p/q-lashings.

lashing = グルグル巻き？



T : a once-punctured torus P : a twice-puctured disk

$$\begin{array}{rcl} T & \simeq & P \\ T & \not\cong & P \\ T \times I & \cong & P \times I \quad =: \mathcal{H}_2 \\ K(a,b) \subset T \times \{0\} \subset T \times I & \cong & P \times I \supset P \times \{0\} \supset C_\nu, C_\mu, C_\lambda \end{array}$$

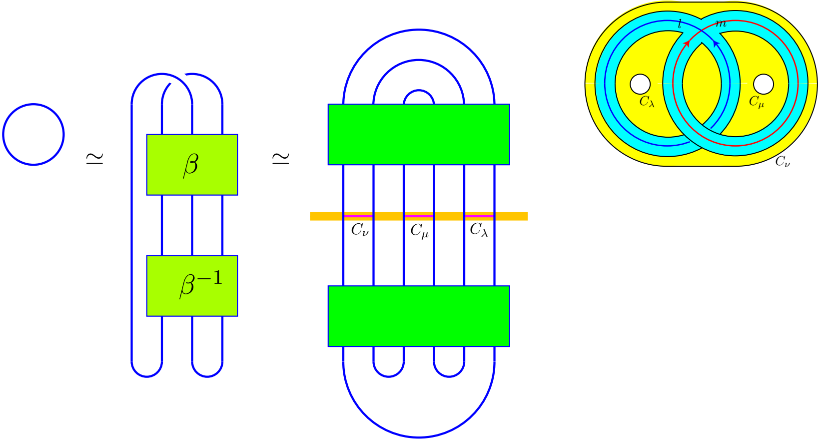
The knot $K(a,b) \sim a[m] + b[l]$, ie, (a,b)-curve in $T \times \{0\}$.

Lemma (Baker-Leucke, Lemma2.2)

$(K(a,b); ab) \cong \left(C_\nu, C_\mu, C_\lambda \ ; \ +1, -\frac{b}{a+b}, -\frac{a}{a+b}\right)$ *as a surgery in* $\mathcal{H}_2$.

Lemma : $(K(a,b); ab) \cong (C_\nu, C_\mu, C_\lambda \;;\; +1, -\frac{b}{a+b}, -\frac{a}{a+b})$

The method of construction:



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ご清聴 ありがとうございました.