「Four Dimensional Topology」 '08 Feb. 7, Hiroshima Univ.

Torus knots, Generalized rational blow-down, and lens space surgery of Type 7,8

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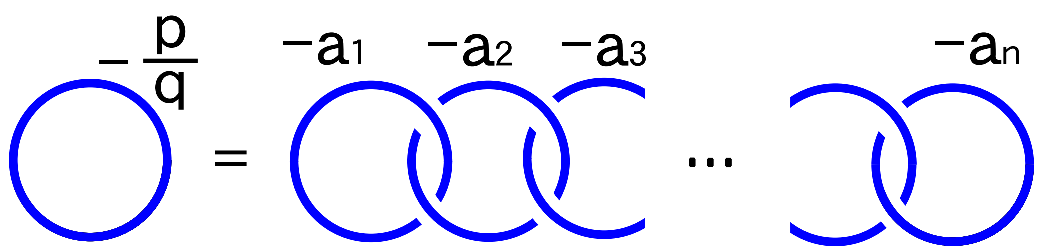
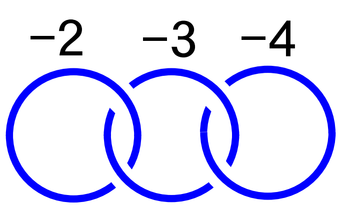
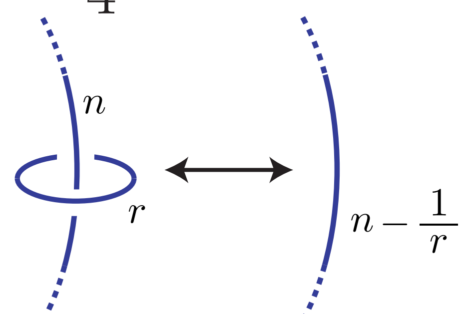
Framed Links for Lens $L(p,q)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$



$L(18,11)$ $(= L(18,5))$ $\quad \frac{18}{11} = 2 - \cfrac{1}{3 - \cfrac{1}{4}}, \quad \frac{18}{5} = 4 - \cfrac{1}{3 - \cfrac{1}{2}}$



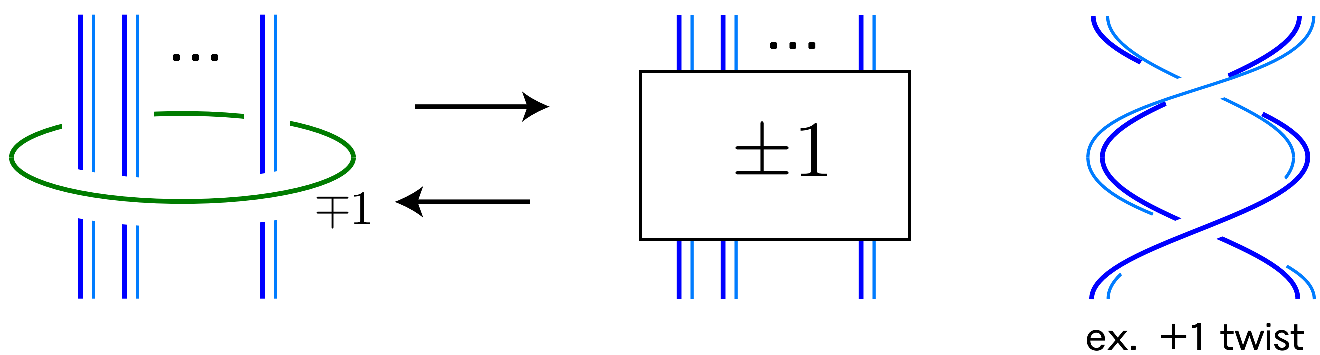


For $n \in \mathbf{Z}, r \in \mathbf{Q}$

Thm. Kirby Calclus ([Fenn-Rourke] ver.)

Framed links L_1, L_2 are moved to each other as below and isotopy,

$\Leftrightarrow$ $M^3(L_1) \cong M^3(L_2)$.



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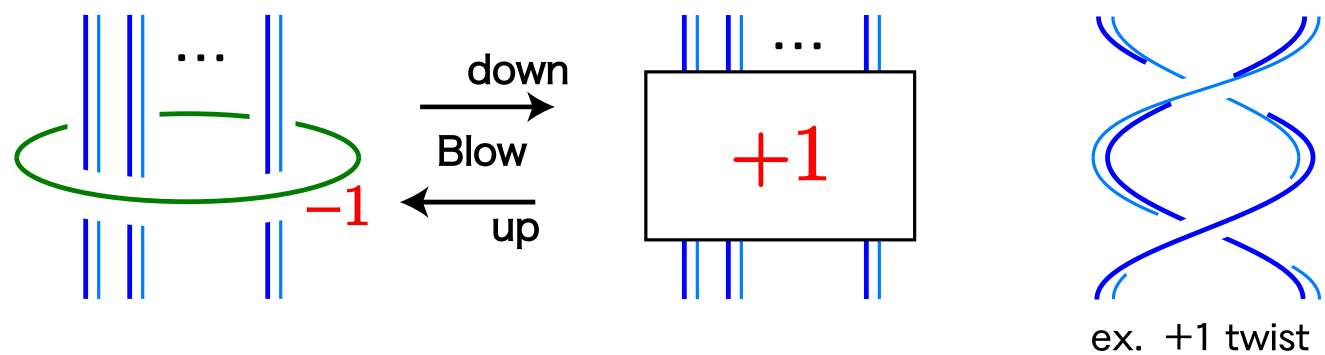
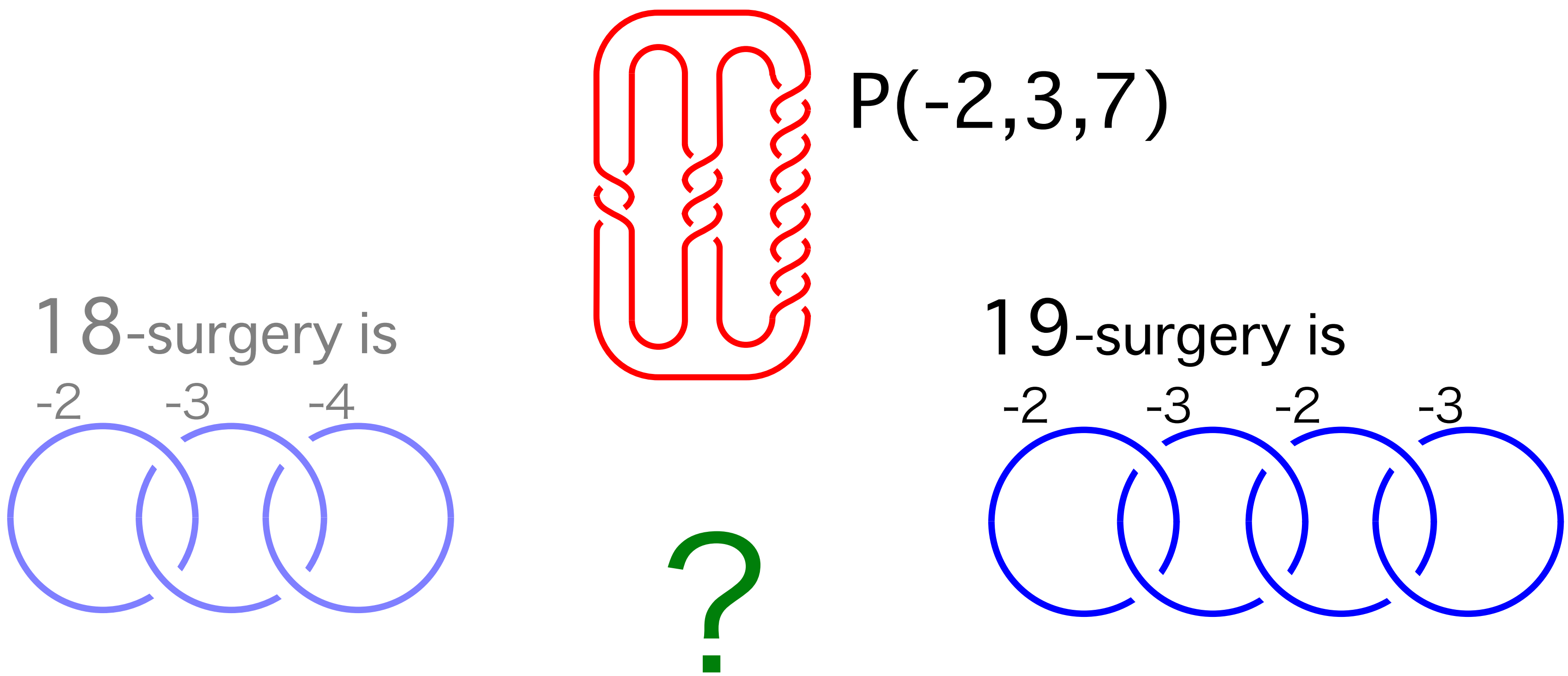


That with this sign is Blow-up/down, related to resolution of singularity.

Subject 1. : Lens space surgery

Unexpected Example [Fintushel-Stern'80]



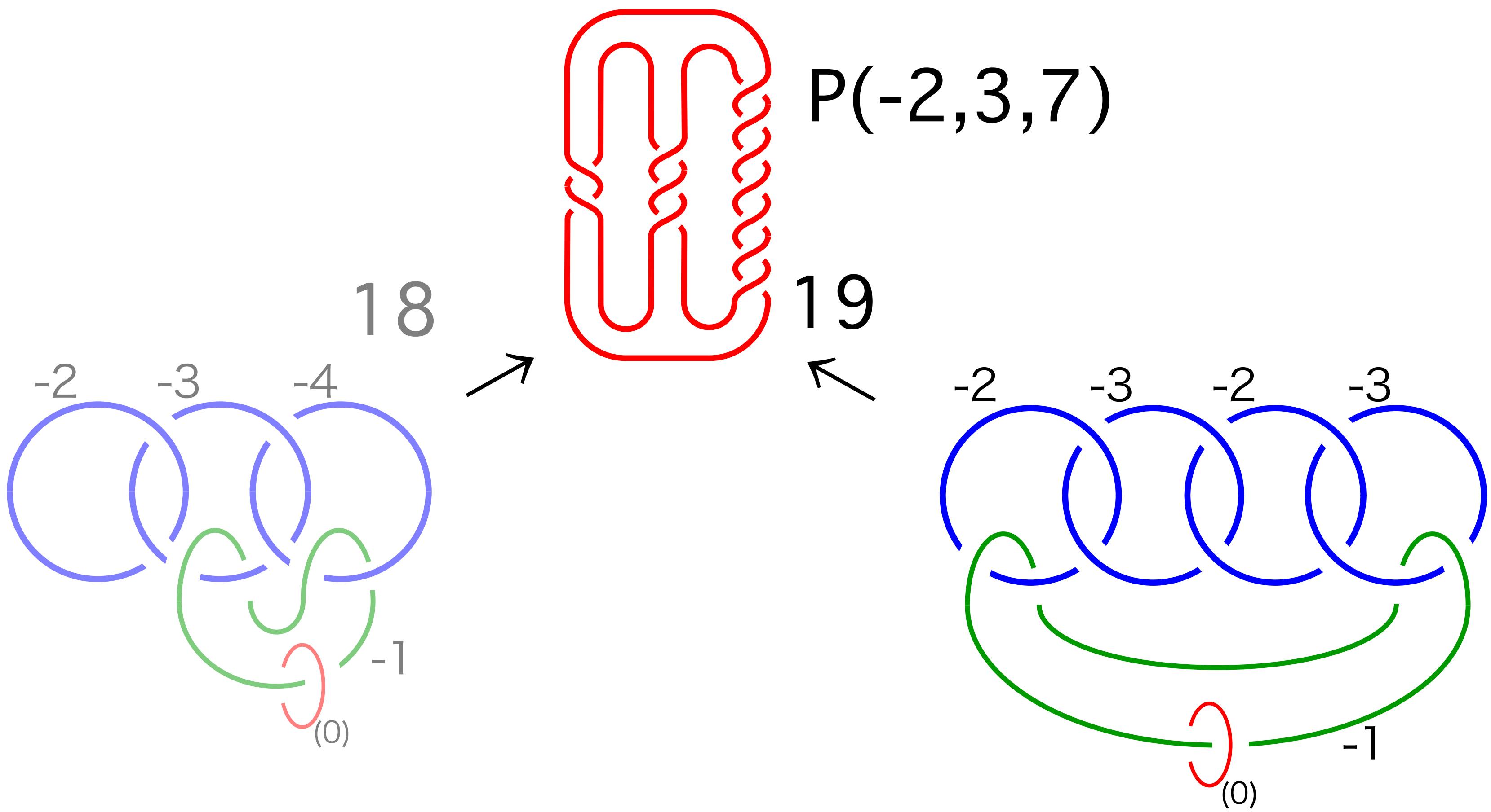
"Why" and "How many" do such examples exists?

$\Rightarrow$ lens space surgery.

My research What is *the best Kirby calc.* to prove it?

One answer ([Y]) :

blue $\cup$ green $= S^3$, and red becomes the knot.



Only Blow downs! ($\Rightarrow$ Resolution of singularity).

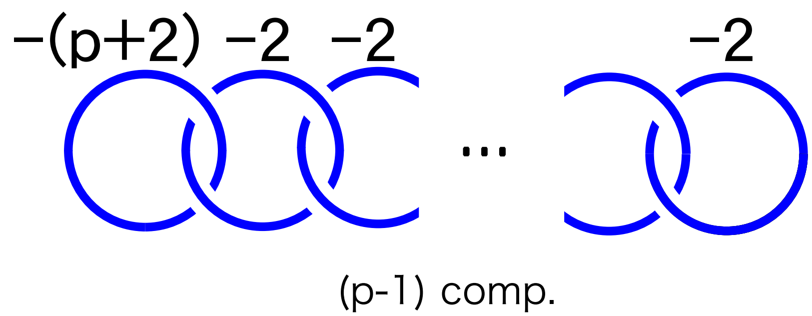
Subject 2. : generalized Rationa blow down (gRBD)

[Fintushel–Stern, 1997]

Log-transform. (on an elliptic surface) can be realized by $(p-1)$ blow-ups and Rational blow down:

Surgery along $L(p^2, p-1)$.

cutting the plumbed negative 4-manifold C_p, and paste a rational homology ball B_p. $(\pi_1(B_p) = \mathbf{Z}/p\mathbf{Z})$



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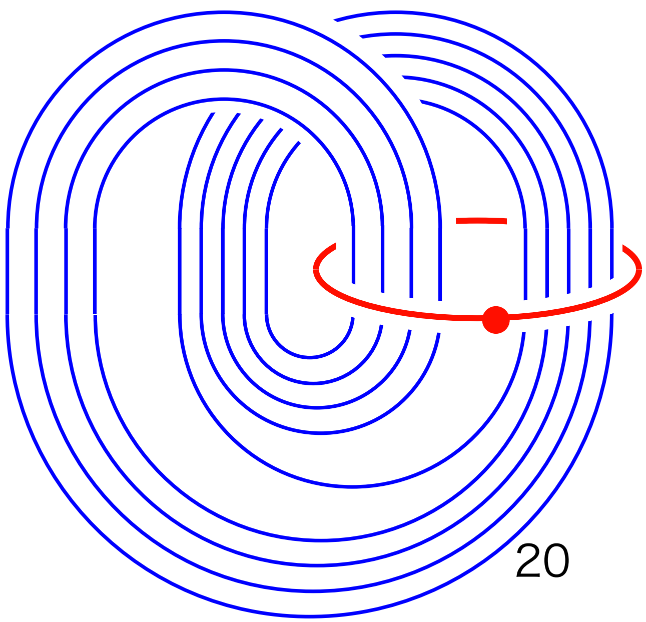
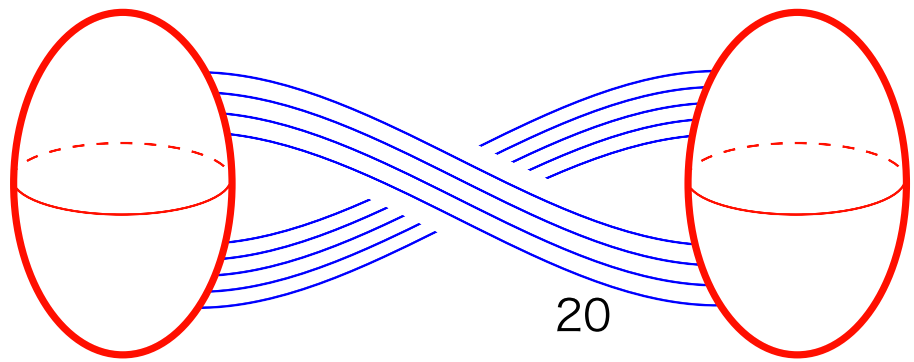
[J.Park,1997] Surgery along $L(p^2, pq-1)$. $\gcd(p,q) = 1$.
cutting the plumbed negative 4-manifold $C_{p,q}$, and
paste a rational homology ball $B_{p,q}$

$\Rightarrow$ Many *exotic* 4-manifolds with "low β_2" are constructed.

My research: I want to see *the best Kirby diagram* of $B_{p,q}$,
following the construction [Casson–Harer,1981] by Kirby calc.

Ex. $(p, q) = (9, 2)$ ($\Rightarrow (a, b) = (4, 5)$ by Euclidean Algorithm)





Rationa homology ball $X_{9,2}$

The blue component is $T(4, 5)$.

$X_{9,2}$ satisfies

$$\pi_1(X_{9,2}) \cong \mathbf{Z}/9\mathbf{Z}, \qquad \partial X_{9,2} \cong L(9^2, 9 \cdot 2 - 1).$$

$X_{p,1} = B_p$, but I do <u>not</u> show $X_{p,q}$ = Casson–Harer's $B_{p,q}$.

Today Talk: Main results

- ∘ Kirby calc. of Type 7,8 ([Y '06])
- ∘ Kirby diagram of gRB $X_{p,q}$ ([Y (Arxiv0708)])

Two subjects :

lens space surgery of Type 7,8 and generalized rational ball

are related to each other,

and also related to Torus knots, and Euclidean Algorithm.

Today Talk: Main results

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Euclidean Algorithm.

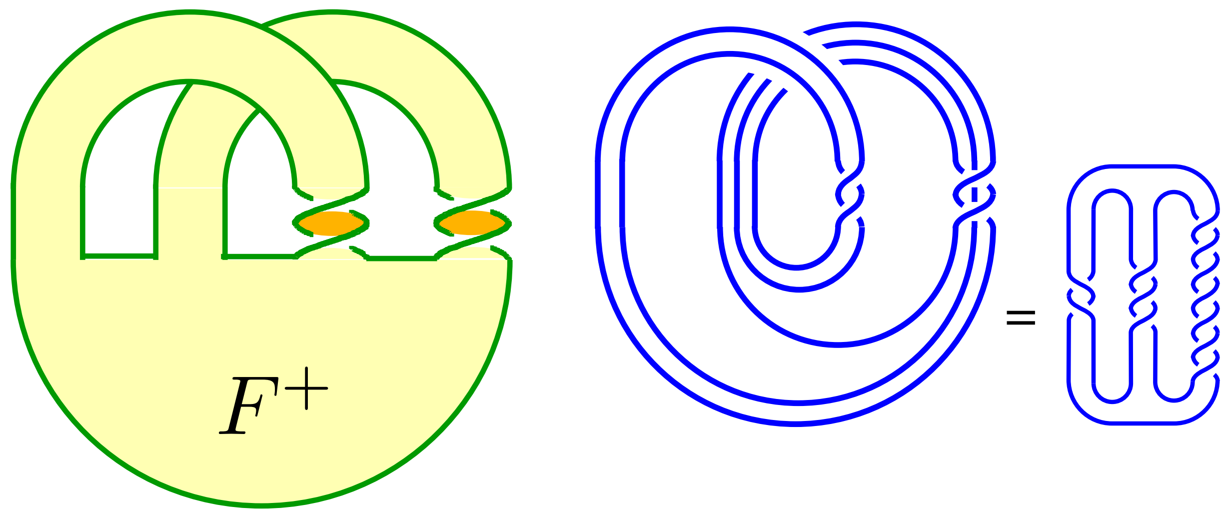
$$(4,5) \to_R (4,1) \to_L (3,1) \to_L (2,1) \to_L (1,1)$$

That is the resolution of the singularity $z^5 - w^4 = 0$.

Subject 1 (detail) : Type 7 ([Berge], see also [Y])

A knot $k^+(a,b)$ in a fiber surface F^+ of left-handed trefoil with p-framing is $L(p,q)$. ($p = a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)

- $k^+(2,3)$ is $Pr(-2,3,7)$. $2^2 + 2 \cdot 3 + 3^2 = 19$.

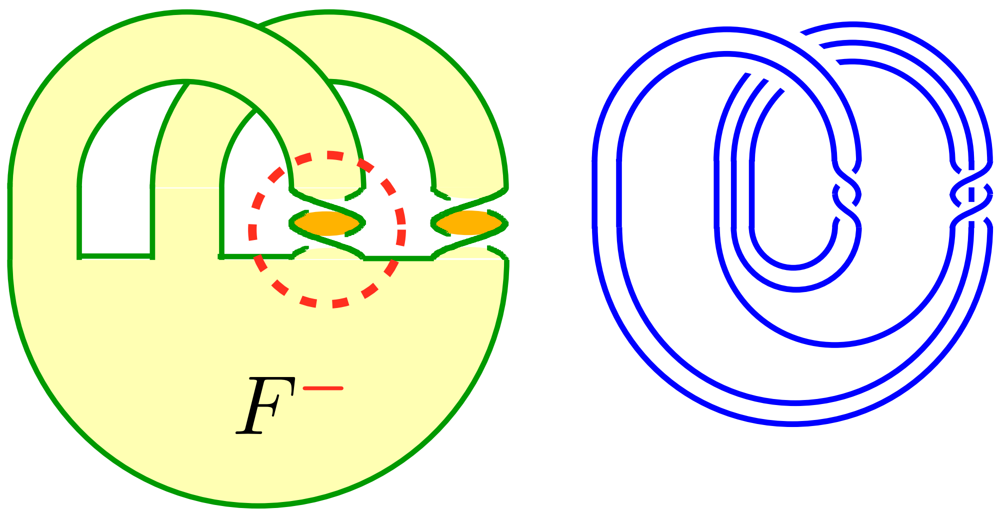


$k^+(a,b)$ is obtained from torus knot $T(a,b)$ by two + full-twists.

Subject 1 (detail) : Type 8 ([Berge], see also [Y])

A knot $k^{-}(a,b)$ in a fiber surface F^{-} of figure8 with p-framing is $L(p,q)$. ($p = -a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)

- $k^{-}(2,3)$ is $T(3,4)$ (unfortunately). $-2^2 + 2 \cdot 3 + 3^2 = 11$.



$k^{-}(a,b)$ is obtained from torus knot $T(a,b)$ by two full-twists, $+$ and $-$.

Subject 1 (detail) :

Type 7,8 is the second family of Berge's Classification of lens space surgeries.

(1) Dehn surgery that keeps Solid torus (Type 1..6 [Berge'91])

Type 1 $= \{\, (ab \pm 1)$-framed $T(a,b)$ is $L(ab \pm 1, -a^2)\ \}$.

(2) Type 7, 8 (knots in genus1 fiber surface)

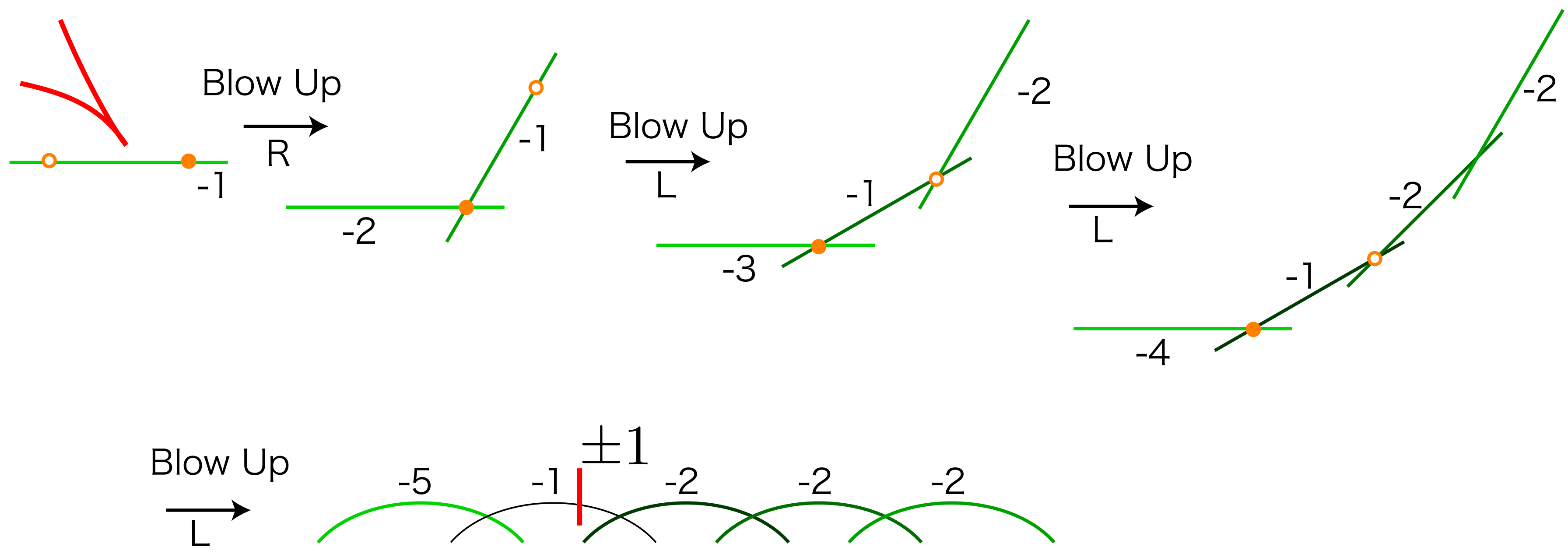
(3) Sporadic examples (cf. M. Tange's work)

Dehn surgery ab-framed $T(a,b)$ is reducible : $-\big(L(a,b)\sharp L(b,a)\big)$.

Tange's work : Some families of lens space surgeries in $\Sigma(2,3,*)$ look like "sporadic examples" in S^3.

[BASE] “Unknot $T(4,5)$ by twist” = “Resolution of $z^5 - w^4 = 0$”.

$$(4,5) \ \to_R \ (4,1) \ \to_L \ (3,1) \ \to_L \ (2,1) \ \to_L \ (1,1)$$

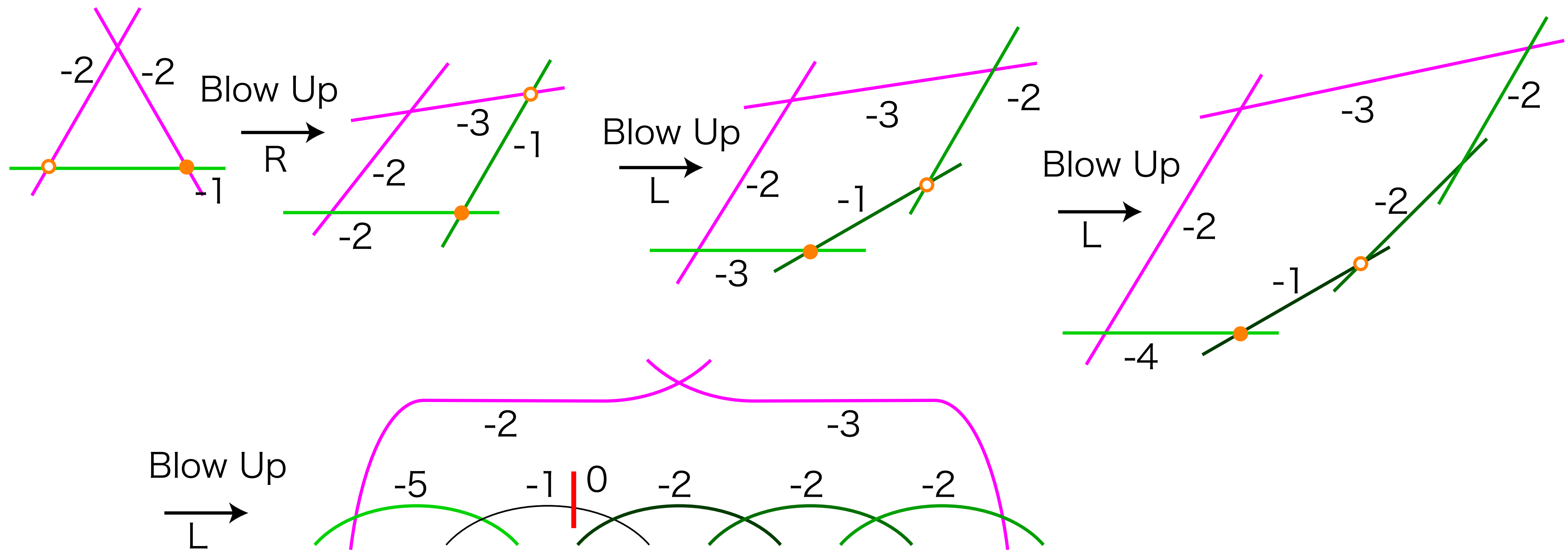


$$[5,2,2,2,2] = \frac{21}{5}, \qquad [5,0,2,2,2] = [7,2,2] = \frac{19}{3}$$

$$(T(4,5);21) = L(21,5), \quad (T(4,5);19) = L(19,3)$$

Proof. $k^+(4,5)$ is obtained from $T(4,5)$ by two components(pink).

$$(4,5) \rightarrow_R (4,1) \rightarrow_L (3,1) \rightarrow_L (2,1) \rightarrow_L (1,1)$$



$$[5,2,3,2,2,2] = \frac{61}{14}$$

$(k^+(4,5);61) = L(61,14)$. $(61 = 4^2 + 4 \cdot 5 + 5^2)$

Subject 2 (detail) : Euclidean Algorithm

For given pair (p, q) with $\gcd(p, q) = 1$ and $p > q$,
we get (a, b) as follows:

$$(p - q, q) \ \rightarrow \cdots \ (\rightarrow_R \ \text{or} \ \rightarrow_L) \cdots \ \rightarrow \ (1, 1)$$

Ex. $(p, q) = (9, 2)$:

$$(7, 2) \ \rightarrow_L \ (5, 2) \ \rightarrow_L \ (3, 2) \ \rightarrow_L \ (1, 2) \ \rightarrow_R \ (1, 1)$$

Note that $a + b = p$. If $q = 1$, then $(a, b) = (p - 1, 1)$

Then following Kirby diagram $T(a, b) \cup U$ (We call it $X_{p,q}$.)
($T(a, b)$ is ab-framed, and U is a 1-handle)
represents...

Subject 2 (detail) : Euclidean Algorithm

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$$\begin{array}{llllll} (p-q, q) & \to \cdots & (\to_R \text{ or } \to_L) \cdots & \to & (1,1) \\ (1,1) & \to \cdots & (\to_R \text{ or } \to_L) \cdots & \to & (a,b) \end{array}$$

Ex. $(p, q) = (9, 2)$

$$\begin{array}{lllllllll} (7,2) & \to_L & (5,2) & \to_L & (3,2) & \to_L & (1,2) & \to_R & (1,1) \\ (1,1) & \to_L & (2,1) & \to_L & (3,1) & \to_L & (4,1) & \to_R & (4,5) \end{array}$$

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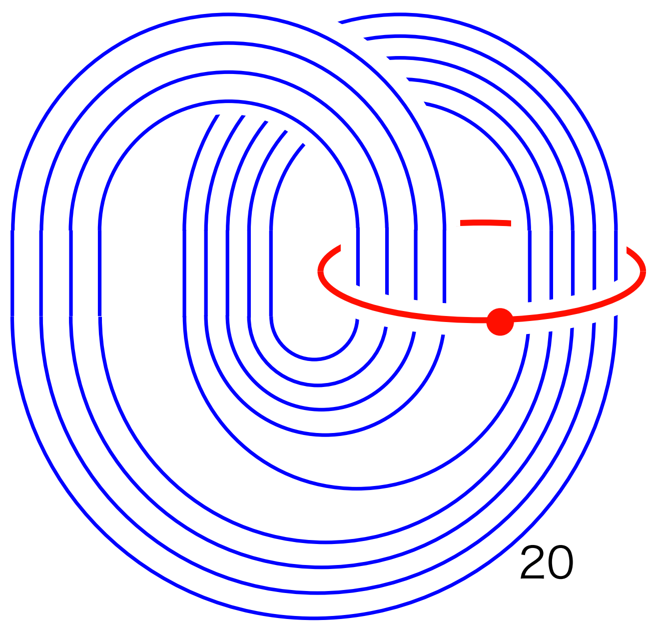
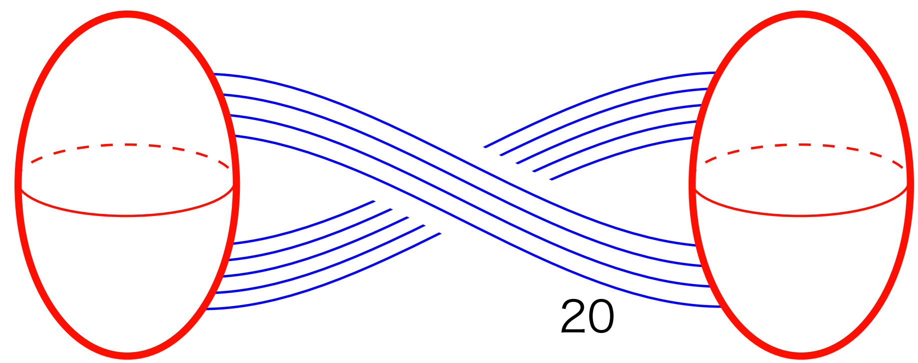
Rational homology ball $X_{p,q}$ satisfing

$$\pi_1(X_{p,q}) \cong \mathbf{Z}/p\mathbf{Z}, \qquad \partial X_{p,q} \cong L(p^2, pq-1).$$

requied condition for generalized rational blow down.

Ex. $(p,q) = (9,2)$, ($\Rightarrow (a,b) = (4,5)$ by Euclidean Algorithm)





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Dr. K. Yasui informed me:

Even if $X_{p,q}$ is not diffeo. to $B_{p,q}$, calculus on Seiberg-Witten class on the result of gRBD on M^4 is same (in some cases).
The resulting 4-manifolds are homeo. and have same SW.

$$(M^4 \backslash C_{p,q}) \cup X_{p,q} \approx (M^4 \backslash C_{p,q}) \cup B_{p,q}$$

$$SW\big((M^4 \backslash C_{p,q}) \cup X_{p,q}\big) = SW\big((M^4 \backslash C_{p,q}) \cup B_{p,q}\big)$$

(by [Michalogiorgaki]).

[BASE] "Unknot $T(4,5)$ by twist" $=$ "Resolution of $z^5 - w^4 = 0$".

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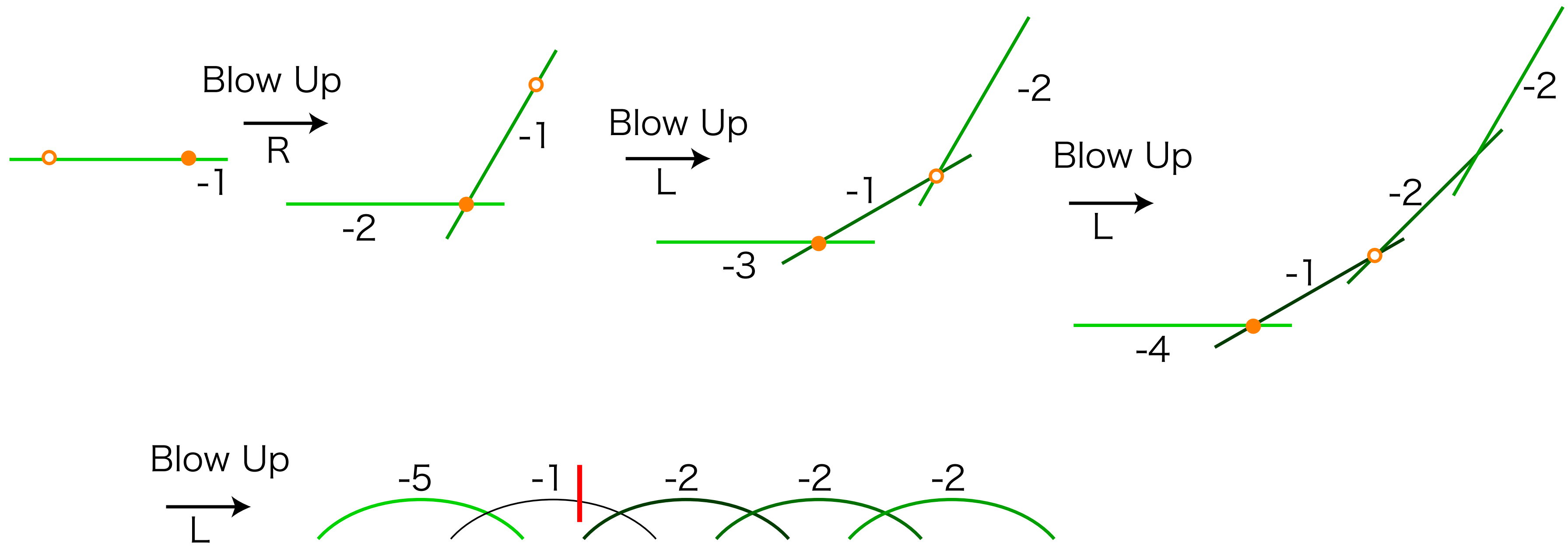
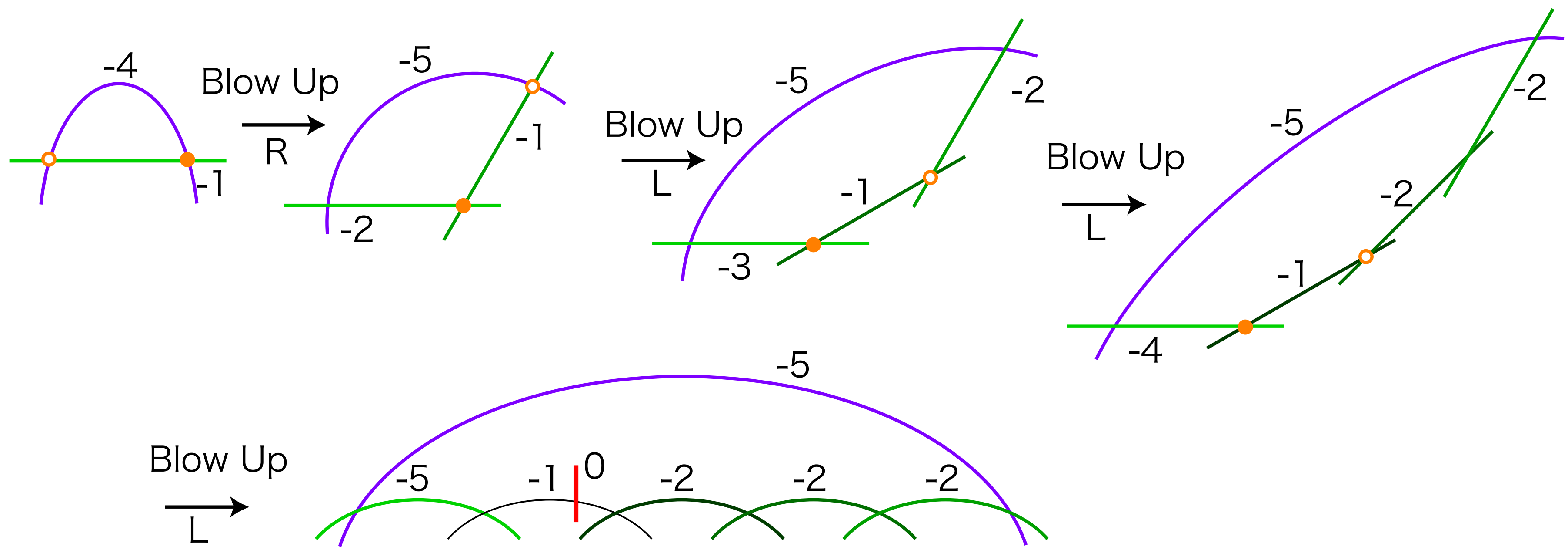


Proof. Link $T(4,5) \cup U$ consists of $T(4,5)$ and a component(violet).

$$(4,5) \ \to_R \ (4,1) \ \to_L \ (3,1) \ \to_L \ (2,1) \ \to_L \ (1,1)$$

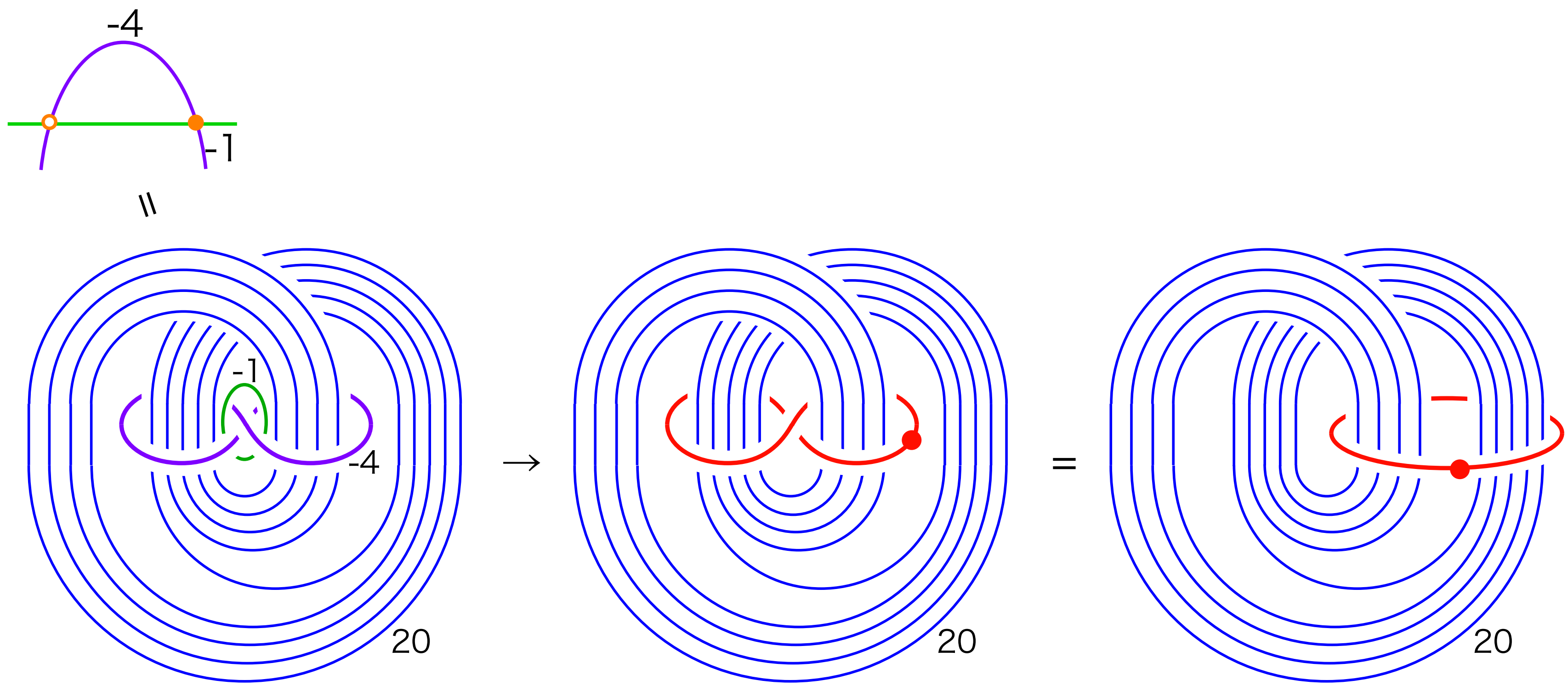


$$[5,5,2,2,2] = \frac{81}{17}$$

$\partial X_{9,2} = L(81,17)$. $(81 = 9^2 \ \ 17 = 9 \times 2 - 1)$

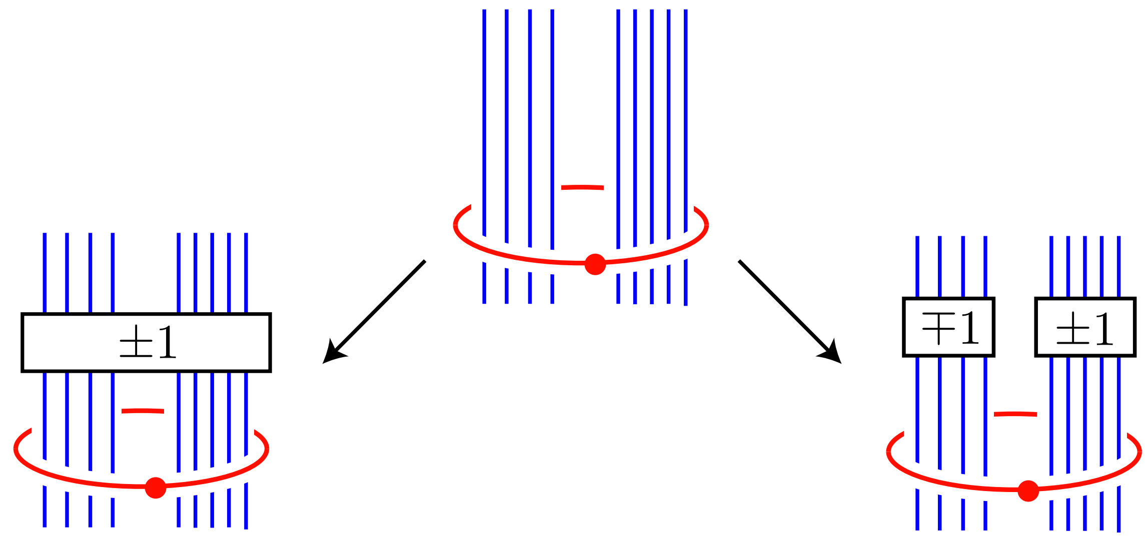
Here, we use Kirby Calc. including surgery also

surgery = " remove $S^2 \times D^2$ paste $D^3 \times S^1$ "

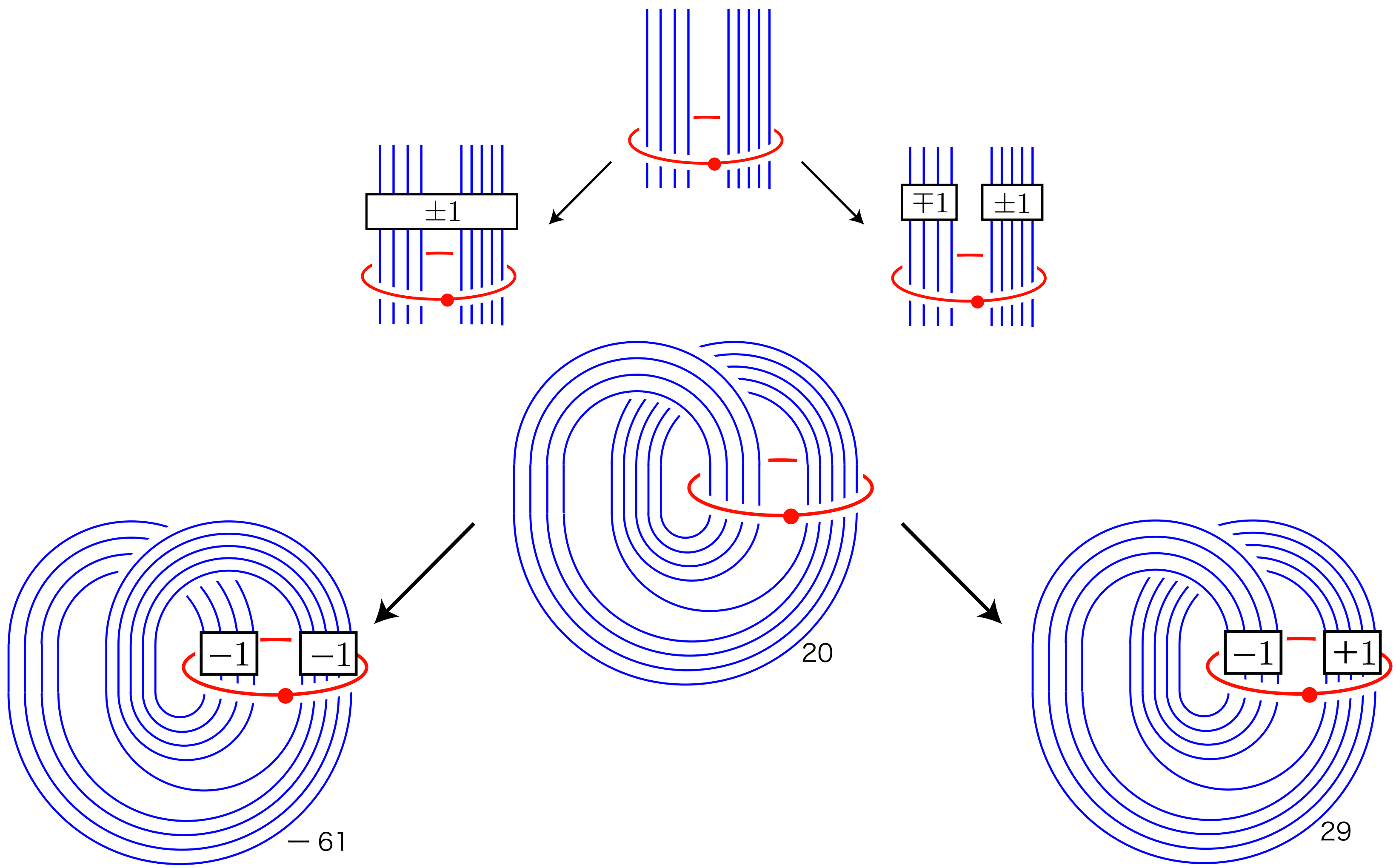


A formula on 1-handle. (not so famous)

$\sim$ *changing of identification* of $S^1 \times D^3$ with $h^0 \cup h^1$.



Using the formula, $T(a, b)$ component becomes



Type 7 : $k^{+}(a, b)!$

$ab - (a + b)^2$

Type 8 : $k^{-}(a, b)$

$ab - a^2 + b^2$

Final Remarks Dehn surgery on the link $X_{p,q}$

(1) Prof. T. Kadokami checked that

$$\partial X_{5,2} \cong \pm L(25, 9)$$

by Alexander poly.-**Reidemeister Tor.** by using his method.

Prof. N. Maruyama also checked it by her formula on

Casson-Walker's inv. $\lambda_{CW}(L(p^2, pq-1)) = \dfrac{p^2-1}{6p^2}$

Why does not it depend on q?

(2) Prof.s K. Ichihara & T. Saito pointed out that the "mechanisim" for $X_{p,q}$ to yield lens space seems "non-generic" in his sense.
(It seems to be essence of Type 7,8.)

Thank you very much!