Dehn surgery along the Mazur link and Akbulut-Yasui links

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The 11th East Asian School of Knots and Related Topics

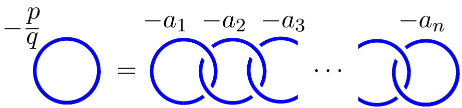
Osaka City University

1. Intro.
2. The Mazur Link and the Akbulut-Yasui Link
3. Using Martelli-Petronio-Roukema's results

A Framed link describes 3-dim and 4-dim manifolds.

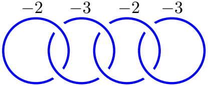
Lens space $L(p, q)$

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$



ex. $L(19, 12)$ $-L(19, 7)$

$$\frac{19}{12} = 2 - \frac{1}{\frac{12}{5}} = 2 - \cfrac{1}{3 - \cfrac{1}{2 - \frac{1}{3}}}$$



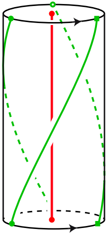
Seifert manifold : 3-manifolds that admit S^1-action without fixed points

Classified by the base surface Σ, obstruction $b \in \mathbb{Z}$ and the sigular fibers $\{(\alpha_i, \beta_i)\}$ (coprime pairs)

Today's Notation $\Sigma(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n))$

- $\Sigma = S$ (S^2), D a disk, or A an annulus.
- Identification b is omitted if $b = 0$.

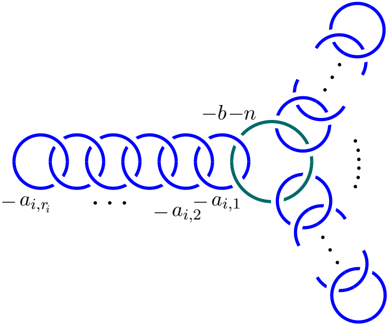
$$
\begin{aligned}
&(b\,; \cdots, (\alpha_i, \beta_i), \cdots) \\
&\quad \sim (b-1\,; \cdots, (\alpha_i, \beta_i+\alpha_i), \cdots) \\
&\quad \sim (b\,; \cdots, (\alpha_i, \beta_i+\alpha_i), \cdots, (\alpha_j, \beta_j-\alpha_j), \cdots) \\
&\quad \Rightarrow b + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \text{ is invariant.}
\end{aligned}
$$

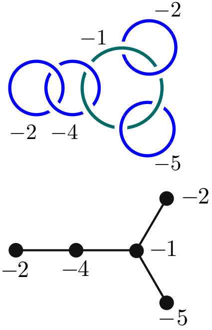


(3,1)

Seifert manifold : 3-manifolds that admit S^1-action

A diagram of $S(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n))$

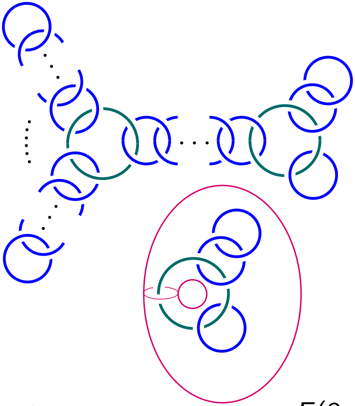


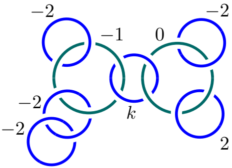


$$\frac{\alpha_i}{\alpha_i - \beta_i} = a_{i,1} - \cfrac{1}{a_{i,2} - \ddots - \cfrac{1}{a_{i,n}}}$$

ex. $S(-2; (2,1), (5,4), (7,5))$

Graph manifold : obtained from some Seifert pieces by pasting along boundary tori





Decomposing torus

$E(3_1)$ $Kb\widetilde{\times}I$

ex. $D((2,1),(3,1)) \cup_{M_k} D((2,1),(2,-1))$

Decomposition of 3**-manifolds** [W. Thurston] Yi Liu's talk

By sphere- and torus- decomposion, any compact orientable 3-manifold can decomposed into pieces :

(A) **Hyperbolic**	(B) Seifert
(1) Toroidal	(0) Reducible

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(A) **Hyperbolic**	(B) Seifert
(1) Toroidal	(0) Reducible

Exceptional Dehn surgery

For a hyperbolic manifold M having a torus boundary, Dehn filling by a solid torus can be non-hyperbolic !

But, it is known that, for M, there exists **finite** such fillings.

Thus, they are called **exceptional Dehn filling/ surgery**.

We may say "Seifert surgery", "Toroidal surgery", and so on.

A solid torus filling is classified by the image of the meridian, called slope $\gamma \subset \partial M$. In the case M is a knot exterior, slopes are parametrized by $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.

Exceptional surgery along two bridge knots

[Brittenham-Wu '01]

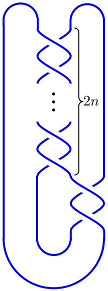
Seifert surgery happens only for (up to mirror image)

$K_{[-2n,2]}$

$n=-1$: Figure eight knot

$n=1$: Right-handed trefoil

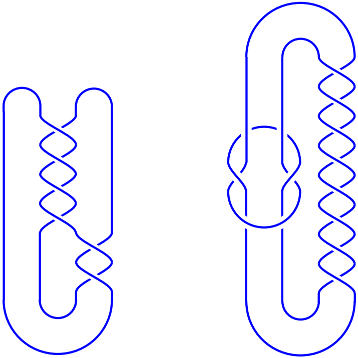
(except Torus knots $T(2,q)$)



- Seifert : slope $1,2,3$, Toroidal : slope $0,4$
- $n=-1$ is special. Seifert : $\pm1,\pm2,\pm3$, Toroidal : $0,\pm4$
- Toroidal : 0-surgery along $K_{[b,c]}$ with odd b and even c

Remark [Akbulut '91] "relative exotic pair"

$$M^3(K, -1) \cong M^3(K', -1) \quad \text{but} \quad X^4(K, -1) \not\cong X^4(K', -1)$$



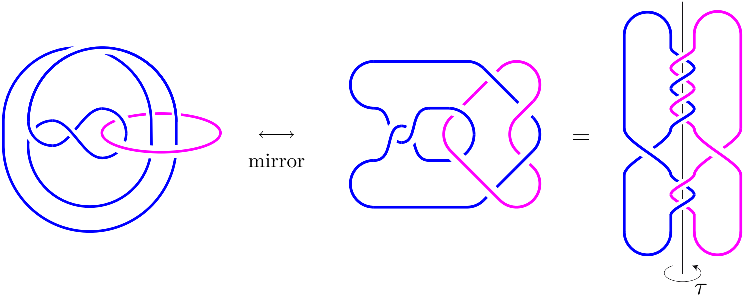
$K = K[-4, 2]$ $\qquad$ $K' = P(-3, 3, -8)$

This K is not slice (not 6_1 in Abe's talk).

The Mazur link

ML = the mirror image of the true* Mazur Link,

hyp. vol. = 4.749...



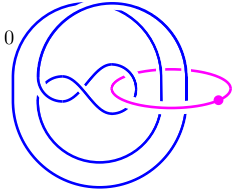
∃ involution τ Symmetry switching components

Not "false" Mazur link by [Prof. Y. Matsumoto]

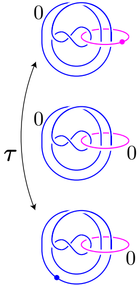
The Mazur manifold[4]

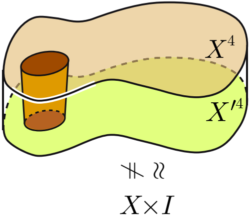
is contractible but not B^4. The boundary $(ML; 0, 0)$ is not S^3

hyp. vol. $= 2.259...$



4-dim 0-, 1-, 2- handles



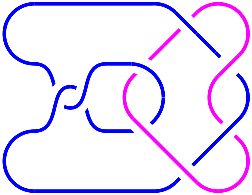


Akbulut Cork $=$ "exoticity maker"

There exsists an 4-dim exotic pair X and X' related to each other by cork twists τ.

Fact ([Y 2014 Nov.])

There exist some lens space (and lens♯lens) surgeries along ML.



$$
\begin{aligned}
(ML; 2, 6) &= L(11, 3)\\
(ML; 2, 7) &= L(13, 8)\\
(ML; 3, 4) &= L(11, 2)\\
(ML; 3, 5) &= L(2, 1)\sharp L(7, 2)\\
(ML; 3, 6) &= L(17, 10)\\
(ML; 4, 4) &= L(3, 2)\sharp L(5, 2)\\
(ML; 4, 5) &= L(19, 8)
\end{aligned}
$$

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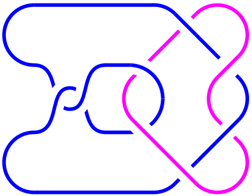

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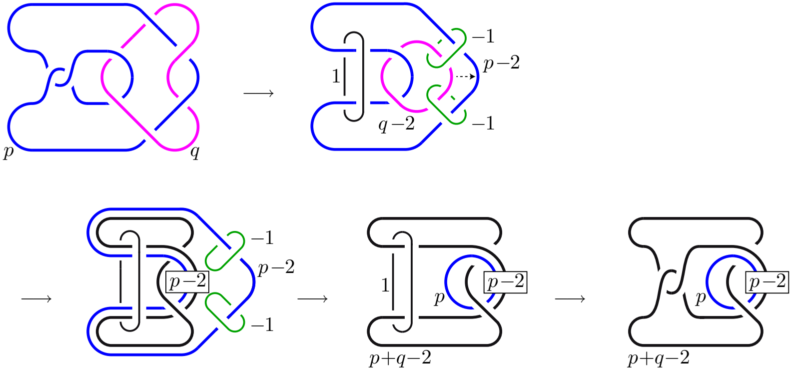
(ML; 4, 5) &= L(19, 8)
\end{aligned}
$$

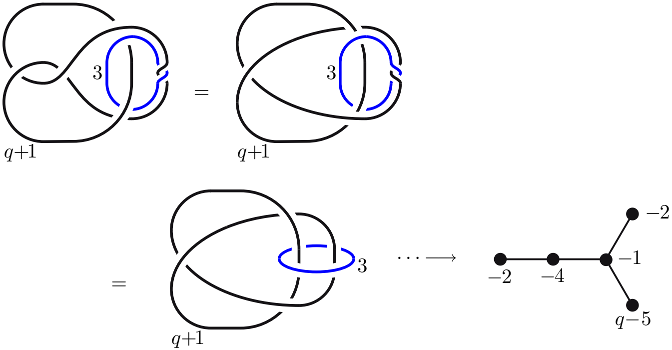
Table 1
ML

	(2, 6)	(2, 7)
(3, 4)	(3, 5)	(3, 6)
	(4, 4)	(4, 5)

Proof of a lens space surgery $(ML; p, q)$



Case $p = 3$



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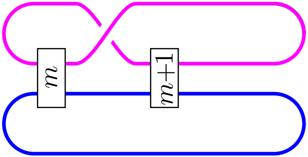
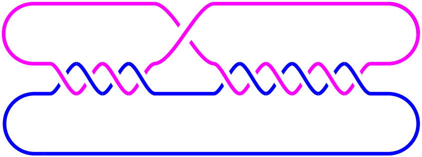
\end{aligned}
$$

The Mazur manifold is generalized to Akubult-Yasui Cork.

How about the **Akubult-Yasui link** AY_m, as a generalization of Mazur Link. It has an involution τ, too.



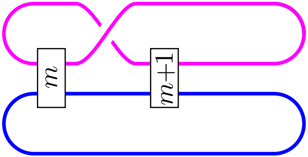
AY_m



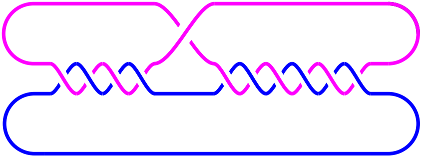
ex. AY_2

Two bridge link $K[2m, 1, 2(m+1)]$

Note that $AY_1 = ML$.



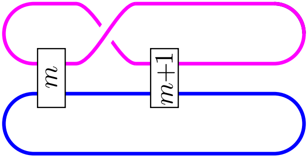
AY_m



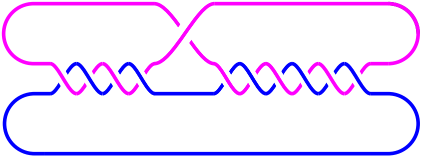
ex. AY_2

Theorem ([Y])

There exist some lens space (and lens♯lens) surgeries along AY_m.



AY_m



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There exist some lens space (and lens♯lens) surgeries along AY_m.

$$
\begin{aligned}
(AY_m; 2m, 2m+4) &= L(4m^2+8m-1, 2m^2+3m-2)\\
(AY_m; 2m+1, 2m+2) &= -L(4m^2+6m+1, 4m+1)\\
(AY_m; 2m+2, 2m+3) &= -L(4m^2+10m+5, 4m+3)\\
(AY_m; 2m+1, 2m+3) &= L(2,1)\sharp - L(2m^2+4m+1, 2m+1)\\
(AY_m; 2m+2, 2m+2) &= -L(2m+1, m)\sharp - L(2m+3, 2)
\end{aligned}
$$

Table 2 Same "pattern" for $m > 1$. Only $m = 1$ is special.

lens♯lens, special cases

$\downarrow p + q = 4m + 4$

ML		$(2,6)$	$(2,7)$
	$(3,4)$	$(3,5)$	$(3,6)$
		$(4,4)$	$(4,5)$
AY_2		$(4,8)$	
	$(5,6)$	$(5,7)$	
		$(6,6)$	$(6,7)$
AY_m		$(2m, 2m+4)$	
	$(2m+1, 2m+2)$	$(2m+1, 2m+3)$	
		$(2m+2, 2m+2)$	$(2m+2, 2m+3)$

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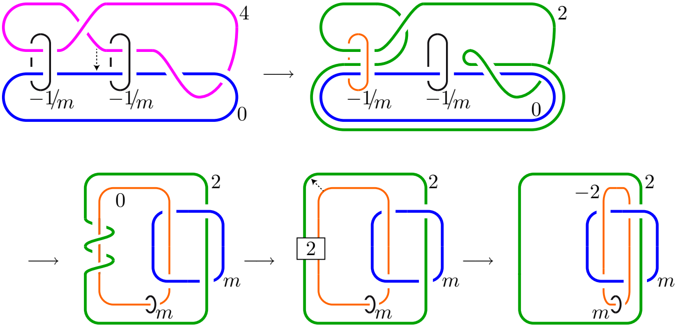
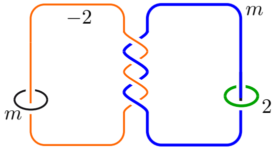
		$(2m+2, 2m+2)$	$(2m+2, 2m+3)$

Conj. [Y, in March 2015]
This is the complete list of lens space surgeries on AY_m.

Proof of the lens space surgery $(AY_m; 2m, 2m+4)$ for any m



$L(4m^2 + 8m - 1, 2m^2 + 3m - 2)$ =



□

Using MPR's results

Martelli-Petronio-Roukema's work :

Exceptional Dehn surgery on the minimally twisted five-chain link,
Comm. Anal. Geom. **22** (2014) no. 4, 689–735.

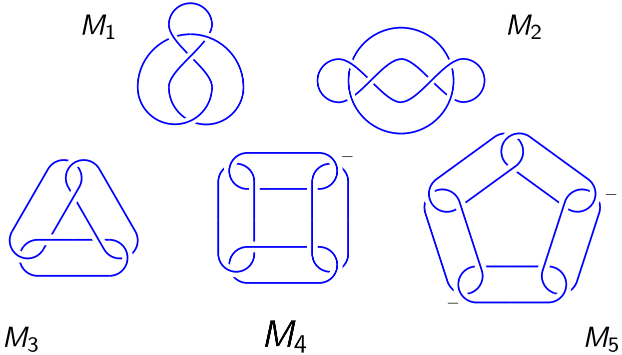
Thanks to Dr. Yuichi KABAYA

and Teragaito's talk on '09

Theorem ([Martelli-Petronio-Roukema '14])

Using computer, Completely decide all exceptional Dehn surgeries along the minimally twisted five-chain link M_5.

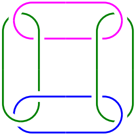
The Minimally Twisted five-Chain link



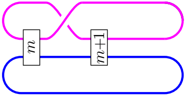
Remark. M_{n+1} is obtained from M_n by blow-up.

M_4 : 8^4_2 : Minimal volume among 4 cusp hyp. [Yoshida '13]
$M_3 = Pr(2,2,2)$, the magic link [Kin-Kojima-Takasawa '13]
M_2 : the Whitehead link, M_1 : the figure-eight knot 4_1.

We can study $AY_m(p, q)$ by M_4



M_4



AY_m

$$AY_m = M_4 \left(-\frac{1}{m}, \emptyset, -\frac{1}{m+1}, \emptyset\right)$$

and thus

$$AY_m(2m + a, 2m + b) = M_4 \left(-\frac{1}{m}, a - 1, -\frac{1}{m+1}, b - 1\right)$$

MPR's SIX MOVES on $(M_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $\quad (\alpha_i \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\})$

Lem. [MPR] These moves do not change (up to mirror image) the manifold.

$$(1) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_4, \alpha_1, \alpha_2, \alpha_3)$$

$$(2) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_4, \alpha_3, \alpha_2, \alpha_1)$$

$$(3) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto \left(\frac{\alpha_1 - 2}{\alpha_1 - 1}, \frac{\alpha_2 - 2}{\alpha_2 - 1}, \frac{\alpha_3 - 2}{\alpha_3 - 1}, \frac{\alpha_4 - 2}{\alpha_4 - 1}\right)$$

$$(4) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto \left(2 - \alpha_1, \frac{\alpha_2}{\alpha_2 - 1}, 2 - \alpha_3, \frac{\alpha_4}{\alpha_4 - 1}\right)$$

$$(5) \quad (-1, \alpha_2, \alpha_3, \alpha_4) \mapsto (-1, \alpha_3 - 1, \alpha_2 + 1, \alpha_4)$$

$$(6) \quad (-1, -2, -2, \alpha) \mapsto (-1, -2, -2, -\alpha - 4)$$

Theorem ([Martelli-Petronio-Roukema '14])

"$(M_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *is non-hyperbolic*"
$\Leftrightarrow$ $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *is in the list below, up to six moves (1)...(6)*

No.	coef.	Manifolds
(O)	$(0, a/b, c/d, e/f)$	$D((b, b-a), (f, f-e)) \cup_H D((2,1), (c-2d, d))$
(A)	$(\infty, a/b, c/d, e/f)$	$S((a,b), (d,-c), (e,f))$
(B)	$(-1, -2, -1, a/b)$	$A((b,-a))/_H$
(d2)	$(-1,-2,-3,-4)$	$D((2,1),(3,1)) \cup_{M2} D((2,1),(2,-1))$
(d3)	$(-1,-3,-2,-3)$	$D((2,1),(3,1)) \cup_{M3} D((2,1),(2,-1))$
(d4)	$(-2,-2,-2,-2)$	$D((2,1),(3,1)) \cup_{M4} D((2,1),(2,-1))$

Pasting matrices : $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Mk = \begin{pmatrix} -1 & k \\ 1 & -(k-1) \end{pmatrix}$, by using a base system {Section, Fiber} of the boundary torus. $|(1,2)\text{-entry}|$ is geometrically well-defind : Intersecton number of the fibers.

Contemporal !

According to Masai's talk in May '15,

The MPR used the computer program SnapPea, thus their proof on hyperbolicity may be not enough, strictly. But, this problem is solved by

Verified computations for hyperbolic 3-manifolds, Exp. Math.

N. Hoffman, K. Ichihara, M. Kashiwagi,
H. Masai, S. Oishi, and A. Takayasu

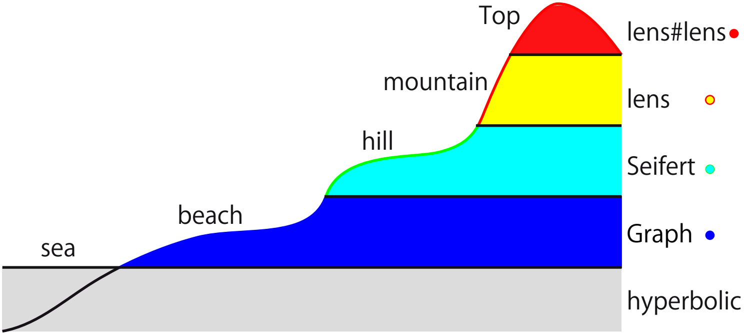
Computer program HIKMOT supported by computer science.

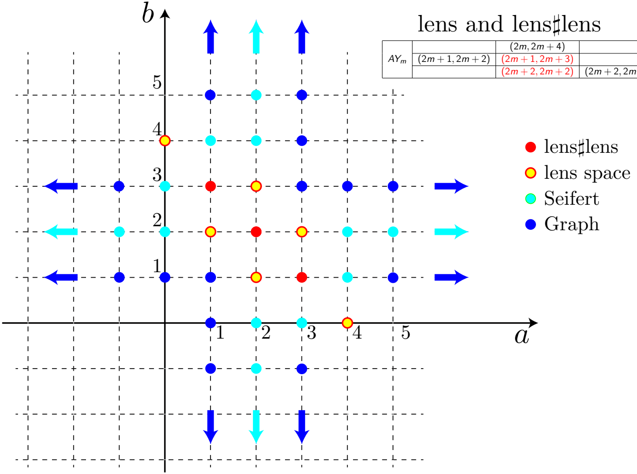
If HIKMOT says "it is hyperbolic", then it is hyperbolic.

The existence of a solution of hyperbolic structure eq. is verified.

Main Results

Geography of Dehn surgeries along AY_m

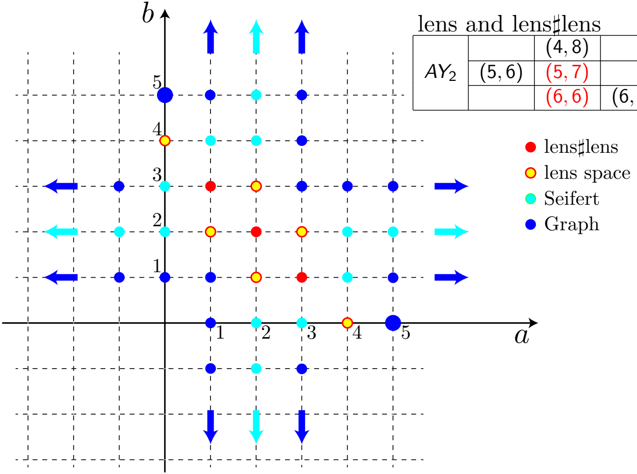




lens and lens♯lens

		$(2m, 2m+4)$	
AY_m	$(2m+1, 2m+2)$	$(2m+1, 2m+3)$	
		$(2m+2, 2m+2)$	$(2m+2, 2m+3)$

Dehn surgery $AY_m(2m+a, 2m+b)$ in general, ie $m \geq 3$.



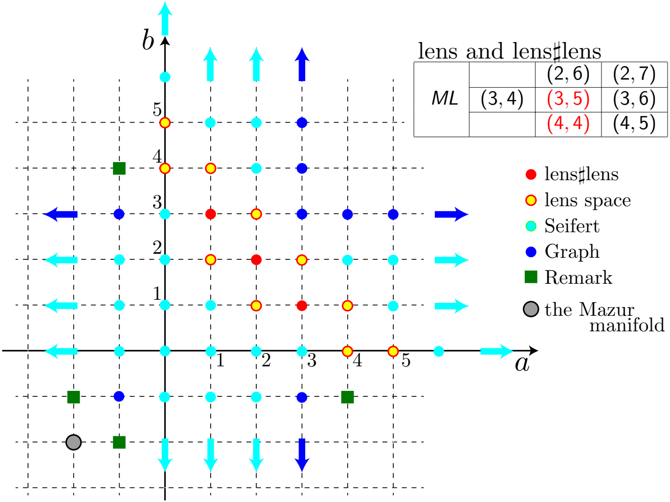
lens and lens♯lens

		$(4,8)$	
AY_2	$(5,6)$	$(5,7)$	
		$(6,6)$	$(6,7)$

Dehn surgery $AY_m(2m+a, 2m+b)$ with $m = 2$.

lens and lens♯lens

		$(2,6)$	$(2,7)$
ML	$(3,4)$	$(3,5)$	$(3,6)$
		$(4,4)$	$(4,5)$

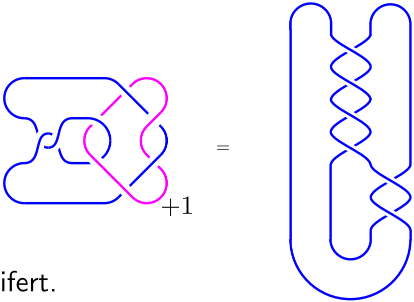


Dehn surgery $ML(2+a, 2+b)$ as $AY_m(2m+a, 2m+b)$ with $m=1$

Remark The hardest case: $a = -1$

The knot $ML(+1, \emptyset) = K_{[-4,2]}$ and $ML(+1, 2+b) = (K_{[-4,2]}, 1+b)$.

Thus we can use Brittenham-Wu's results on two bridge knots.



$b = 0, 1, 2 \Rightarrow$ Seifert.

$b = -1, 3 \Rightarrow$ Troidal.

$b = -1 : (K_{[-4,2]}, 0) = ML(1,1) = M_4(-1, -2, -1/2, -2)$
$= M_4(-1, -3/2, -1, -2) = A((2,3))/_H$

$b = -2 : (K_{[-4,2]}, -1)$ (relative exotic pair to $(P(-3,3,8); -1)$)

[Again]

SIX MOVES on $(M_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $(\alpha_i \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\})$

Lem. [MPR] These moves do not change (up to mirror image) the manifold.

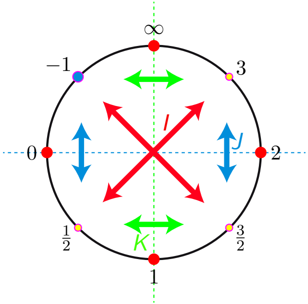
$$
\begin{array}{lll}
(1) & (\alpha_1, \alpha_2, \alpha_3, \alpha_4) & \mapsto (\alpha_4, \alpha_1, \alpha_2, \alpha_3) \\
(2) & (\alpha_1, \alpha_2, \alpha_3, \alpha_4) & \mapsto (\alpha_4, \alpha_3, \alpha_2, \alpha_1) \\
(3) & (\alpha_1, \alpha_2, \alpha_3, \alpha_4) & \mapsto \left(\frac{\alpha_1 - 2}{\alpha_1 - 1}, \frac{\alpha_2 - 2}{\alpha_2 - 1}, \frac{\alpha_3 - 2}{\alpha_3 - 1}, \frac{\alpha_4 - 2}{\alpha_4 - 1} \right) \\
(4) & (\alpha_1, \alpha_2, \alpha_3, \alpha_4) & \mapsto \left(2 - \alpha_1, \frac{\alpha_2}{\alpha_2 - 1}, 2 - \alpha_3, \frac{\alpha_4}{\alpha_4 - 1} \right) \\
(5) & (-1, \alpha_2, \alpha_3, \alpha_4) & \mapsto (-1, \alpha_3 - 1, \alpha_2 + 1, \alpha_4) \\
(6) & (-1, -2, -2, \alpha) & \mapsto (-1, -2, -2, -\alpha - 4)
\end{array}
$$

Moves (3), (4)

$$I(x) = \frac{x-2}{x-1}, \quad J(x) = \frac{x}{x-1}, \quad K(x) = 2 - x,$$

$$I,\ J,\ K : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$$

are involutions and satisfies a relation:



$$I^2 = J^2 = K^2 = IJK = 1$$

Move (5)

$$(-1, \alpha_2, \alpha_3, \alpha_4) \mapsto (-1, \alpha_3 - 1, \alpha_2 + 1, \alpha_4)$$

is also an involution. We only have to care

$$\begin{matrix} (-1, & \alpha_2, & \alpha_3, & \alpha_4) \\ (-1, & \alpha_3 - 1, & \alpha_2 + 1, & \alpha_4) \\ (-1, & \alpha_2, & \alpha_4 + 1, & \alpha_3 - 1) \end{matrix}$$

and their orbits by dihedral moves (1), (2).

TAKE CARE :

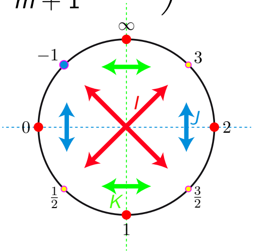
the case **new** -1 **occurs** by Moves (3), (4) (ie, I, J, K).

Method： Once ignore $m=1$ (The Mazur link), the special case.

$$AY_m(2m+a, 2m+b) = M_4\left(-\frac{1}{m},\ a-1,\ -\frac{1}{m+1},\ b-1\right)$$

($a=0$) By Move (5),

$$= M_4\left(-1,\ -\tfrac{1}{m},\ b-1,\ -\tfrac{1}{m+1}\right)$$
$$= M_4\left(-1,\ b-2,\ \tfrac{m-1}{m},\ -\tfrac{1}{m+1}\right)$$
$$= M_4\left(-1,\ -\tfrac{1}{m},\ \tfrac{m}{m+1},\ b-2\right)$$



$\Rightarrow$ $m>2$ and $\{b-1, b-2\} \not\subset \{-1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, \infty\}$ is hyp.
$m=2$ or $b=0,1,2,3,4,5$

Harder case $(a=0,\ b=5)$ By Move (4),

$$= M_4\left(-1,\ 3,\ \tfrac{m-1}{m},\ -\tfrac{1}{m+1}\right) = M_4\left(\tfrac{1}{2},\ -1,\ 1-m,\ \tfrac{2m+3}{m+1}\right) \ldots$$

If $m=2$, then ... , else then ...

Thank you very much!