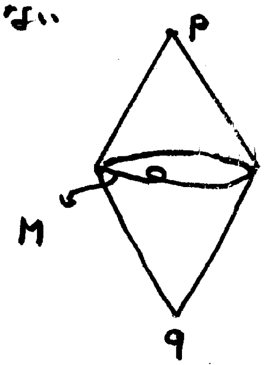
§0 はじめに

・ある空間の topological type を決めたい時、それが「(topological) manifold かどうか」が鍵となることがある。

Example M^3 : a homology 3-sphere $\neq S^3$

ΣM^3 = the suspension of M^3 は manifold でない
(p, q は non-manifold points)



Double Suspension Conjecture :

$$\Sigma(\Sigma M^3) \approx S^5 \quad ?$$

仮に $\Sigma(\Sigma M^3)$ が topological manifold であるとしよう。

すると $\Sigma(\Sigma M^3) \simeq S^5$ だから

Newman の Theorem (top. mfd についての高次元 Poincaré) より

$$\Sigma(\Sigma M^3) \approx S^5$$

Double Suspension Conjecture は正しい (J. W. Cannon, R. D. Edwards)
1978 1975

問題　位相空間 X が n-dim'l (topological) manifold で
あるための (チェックしやすい) 判定条件を与えよ.

Example : $X \approx S^1 \iff$ · X : compact connected metric
· $\forall p \in X$　$X \setminus \{p\}$ connected
· $\forall p \neq q \in X$　$X \setminus \{p, q\}$ disconnected

2-dim'l (topological) manifold の
簡単な判定法がある(はず).　憶い出せません.

問題'　$\mathbb{R}^n$ の判定法 ?

M^n : an n-dim'l topological manifold

- locally compact separable metrizable
- $\dim M^n = n$
- locally contractible
- $H_*(M^n, M^n - x) \cong H_*(D^n, D^n - 0) \cong H_*(D^n, \partial D^n)$
- $f : D^i \to M$　$g : D^j \to M$　$i + j < n$

$\exists f' \doteqdot f$　$\exists g' \doteqdot g$　s.t.　$f'(D^i) \cap g'(D^j) = \phi$

X : locally compact separable metric , locally contractible

$\dim X < \infty$

$H_*(X, X - x) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$

X : a top. manifold $\iff$?

Edwards, Quinn 1970 ~ 198(前)

$n \geq 6$ Daverman の本

Theorem X : a n-dim'l topological manifold $n \geq 5$

$n = 5$ Fund. Math. 2005 ?

$\iff$ (i) $\exists M^n$: a top. mfd

$\exists f : M^n \longrightarrow X$ a cell-like map
あとで

(ii) $\forall f, g : D^2 \to X$ $\forall \varepsilon > 0$

$\exists f', g' : D^2 \to X$ s.t.

$d(f, f') < \varepsilon$ $d(g, g') < \varepsilon$ $f'(D) \cap g'(D) = \phi$

Remark. (i) について. Quinn's index $i(X) \in 8\mathbb{Z} + 1$

(i) $\iff$ $i(X) = 1$

$\Uparrow$

X が manifold point をもつ.

Quinn index $i(X)$ は $n \geq 4$ で 定義され (i) と同値

$n=3$

(i) $\exists M^3 \xrightarrow{\exists f} X$? $\Rightarrow$ 3-dim'l Poincaré

cell-like

(cellular)

Bryant - Ferry - Mio - Weinberger 1995~6? Annals Math.

$i(X) \neq 1$ となる X を構成 $\dim X = 5$

Bing-Borsuk conj.

X : locally compact separable metric locally contractible

$\dim X = n$

$\forall x, y \in X$ $\exists h : X \to X$ a homeo. s.t. $h(x) = y$

$\Downarrow$

X : a top. n-manifold. ?

$n \leq 2$ Bing-Borsuk Ann. Math.

$n=3$ 正しい $\Rightarrow$ 3 dim'l Poincaré conj

(locally)
compact metric X
(separable)

$\dim X \leqq n$: inductive に定義される (Poincaré のアイディア)

$\dim X = -1 \iff X = \phi$

$\dim X = 0 \iff {}^{\forall} F_1, {}^{\forall} F_2$ closed disj ${}^{\exists} U_1, U_2$ disj. open

$$X = \underset{\substack{\cup \\ F_1}}{U_1} \cup \underset{\substack{\cup \\ F_2}}{U_2}$$

space Y の $\dim \leqq n-1$ が defined

$\dim X \leqq n \iff {}^{\forall} F_1, F_2$ disj closed ${}^{\exists} L$ s.t.

$$X \setminus L = \underset{\substack{\cup \\ F_1}}{U_1} \cup \underset{\substack{\cup \\ F_2}}{U_2} \quad \text{disj. open}$$

$$\dim L \leqq n-1$$

X : compact metric

$\dim X \leqq n \iff \forall \varepsilon > 0 \ {}^{\exists} f_\varepsilon : X \to P$ a compact polyhedron ($\dim P \leqq n$)

$$\dim f_\varepsilon^{-1}(x) < \varepsilon \quad \forall x \in P$$

$\wr$

複雑な空間を CW complex で"近似"して調べる.

Ex

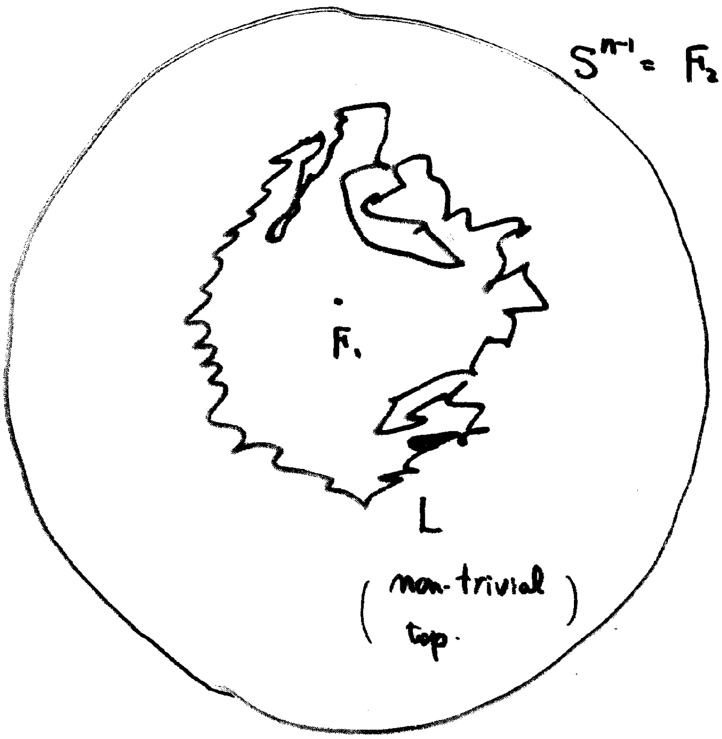
X : hereditarily indecomposable compact conn. metric

$\forall \varphi : [0,1] \to X \qquad \varphi \equiv \text{const}.$

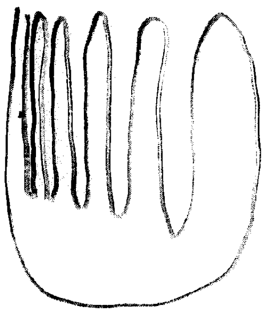
$\rightsquigarrow$

$\pi_q(X) = 0 \quad \forall q \geq 1$ (or 1) singular homology is trivial.

$\mathbb{R}^n$



Ex $\mathbb{R}^2$

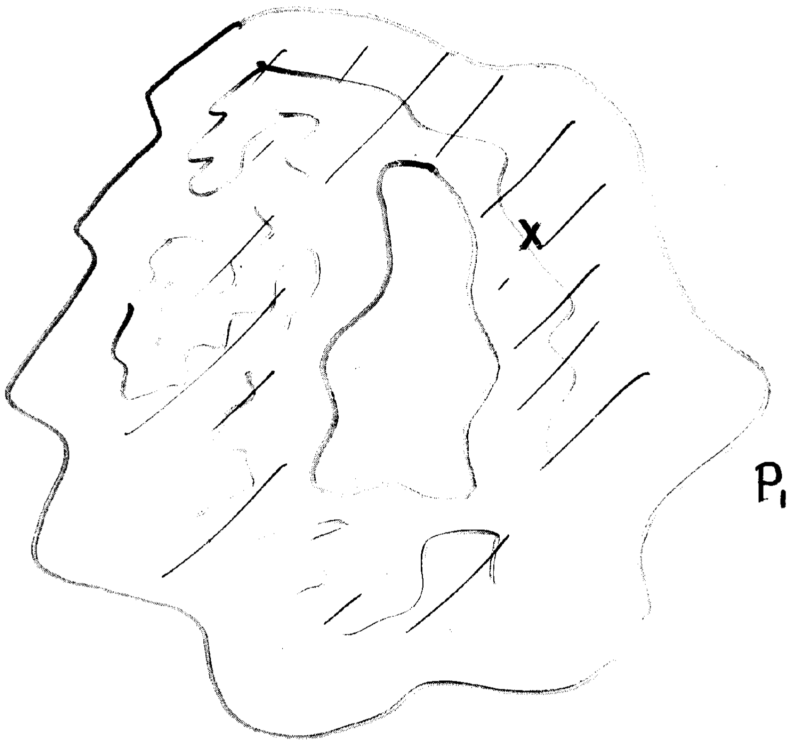


$\pi_q = 0$

nontrivial top.

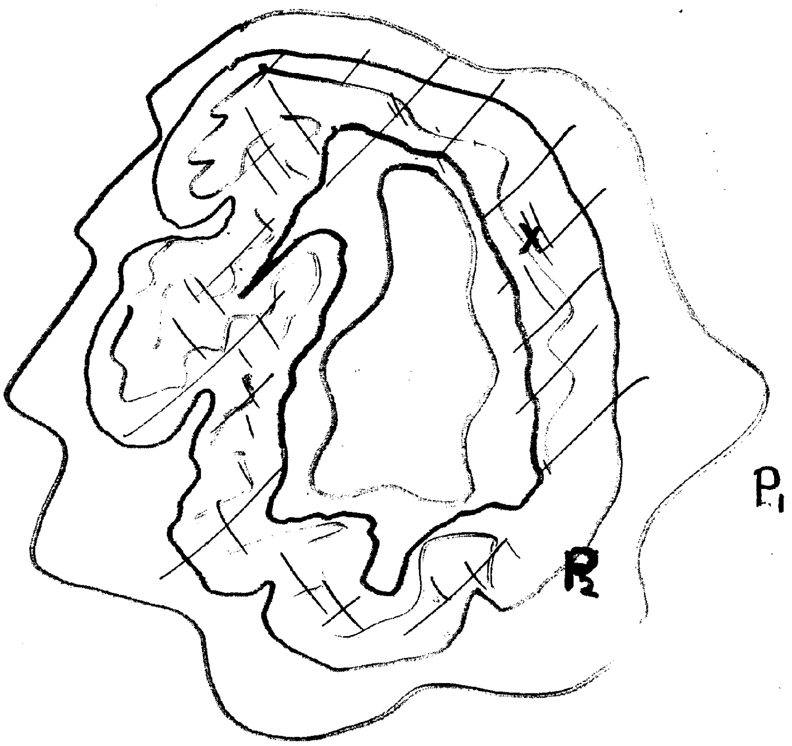
$\dim X < \infty$

$X \hookrightarrow \mathbb{R}^N$



$\dim X < \infty$

$X \hookrightarrow \mathbb{R}^N$



$P_1 \supset P_2 \supset P_3 \supset \cdots \quad \supset \bigcap_{n=1}^{\infty} P_n = X \qquad P_i$: a polyhedron

X を直接みるかわりに　$\{P_i\}_{i=1}^{\infty}$　と

$P_i \overset{f_i}{\hookrightarrow} P_{i-1}$　をみる

P_i , f_i にはふつうの π_q, H_q がうまく働く.

$\check{H}^*(X : G) := \varinjlim H^*(P_i : G)$

o. $\check{H}_*(X : G) := \varprojlim H_*(P_i : G) \qquad \check{\pi}_q(X) = \varprojlim \pi_q(P_i)$
不便

問　M^n : a topological manifold

$f : M^n \to X$: a cell-like map

$\dim X < \infty$?

$n=1$　自明

$n=2$　Moore　$(X \approx M^2)$

$n=3$　正しい　(Walsh)

$n=4$　open　(だと思います)

$n \geq 5$　No.　(Dranishnikov, Dydak-Walsh)

Generalized Schönflies Theorem ([1] p.38)

$i : S^{n-1} \hookrightarrow S^n$ が 次をみたすとする

$$S^{n-1} \times [-1,1] \overset{\exists h}{\hookrightarrow} S^n$$ a topological embedding s.t.

$$h|S^{n-1} \times \{0\} = i$$

$$\Downarrow$$

$S^n \setminus i(S^{n-1})$ の 2つの component の closure は いずれも n-cell と homeo.