Surgery exact sequenceについて

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$\boxed{m \geq 5}$

X : m-dim. Poincaré complex, $\pi := \pi_1 X$

$$L_{m+1}(\mathbb{Z}[\pi]) \to S(X) \xrightarrow{\eta} N(X) \xrightarrow{\theta} L_m(\mathbb{Z}[\pi])$$

manifold structure set

normal str. set

L群

Surgery obstruction

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§1. Manifold str. set $S(X)$

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§3. Surgery obstruction

Reference

A. Ranicki : Algebraic and Geometric Surgery

以下断りのない限り
- 多様体は5次元以上, smooth, oriented, closed
- $S(X) = S^h_{Diff}(X)$
- $N(X) = N_{Diff}(X)$
- $L_*(\mathbb{Z}[\pi]) = L^h_*(\mathbb{Z}[\pi])$

§1. Manifold structure set

Def m-dim oriented Poincaré complex

X : finite CW complex, $\pi := \pi_1 X$

s.t. $\exists [X] \in H_m(X;\mathbb{Z})$
基本類

For $\forall$ $\mathbb{Z}[\pi]$-module Λ

$$[X]\cap : H^*(X;\Lambda) \xrightarrow{\cong} H_{m-*}(X;\Lambda)$$

局所係数

Remark
- closed m-manifold は m-dim. Poincaré cpx
- relative version もある. $(X, \partial X)$

Def X^m : Poincaré cpx

(i) A manifold structure (M, f) on X

$\underset{def}{\Leftrightarrow}$ ・M : closed smooth m-manifold

・$f : M \to X$ ホモトピー同値写像

(ii) Manifold structure set

$$S(X) := \{(M,f)\}/\sim$$

$(M,f) \sim (M',f') \underset{def}{\Leftrightarrow} \exists$ bordism $(F;f,f') : (W;M,M') \to X \times ([0,1];\{0\},\{1\})$

s.t. $(W;M,M')$ は h-cobordism

$\left.\begin{matrix} M \hookrightarrow W \\ M' \hookrightarrow W \end{matrix}\right)$ は h.e.

Remark

- **h-cobordism theorem**

  $\pi_1 M = 1,\ m \geq 5$

  $(W^{m+1}; M^m, M'^m)$: h-cobordism

  $\Leftrightarrow (W; M, M') \underset{\text{diffeo}}{\cong} M \times (I; \{0\}, \{1\})$

- [Freedman]

  4-dim TOP で h-cobordism thm は成立.

- $\pi_1 M \neq 1$ のときは s-cobordism theorem がある

  $\longrightarrow$ Simple homotopy type

  ◎ surgery exact sequence の S版もある

- [Freedman]

  $\pi_1 M$: "Good" ex $\mathbb{Z}$, Abelian, finite, Solvable, ...

  4-dim TOP で s-cobordism thm は成立.

§2 Normal structure

Q M^m : manifold

X^m : Poincaré cpx

写像 $f: M \to X$ はいつホモトピー同値になるか?

$\longrightarrow$ Normal str. & Surgery で調べる

多様体の normal bundle

M^m : closed manifold

$M^m \hookrightarrow S^{m+k}$ $k \gg 1$ (embedding)

N : tublar nbd. of M in S^{m+k}

$\searrow$ normal bdl ν_M の total space と同一視できる

$$S^{m+k} \xrightarrow{\rho} S^{m+k} / (S^{m+k} \setminus N) = N/\partial N = T(\nu_M)$$

($T(\nu_M)$: Thom space)

$$\rho_*[S^{m+k}] = [(N, \partial N)] \in \tilde{H}_{m+k}(N/\partial N) \cong H_{m+k}(N, \partial N)$$

U_{ν_M} : Thom class of ν_M

$\in H^k(N, \partial N)$

このとき

$$\boxed{[M] = U_{\nu_M} \cap [(N, \partial N)]}$$ (Thom 同型)

X^m: Poincaré cpx とする

Spivak normal structure (ν_X, ρ_X) ← normal bundle のホモトピー版

- ν_X: fibration $S^{k-1} \to \nu_X \xrightarrow{p} X$ $(k \gg 1)$
- $\rho_X : S^{m+k} \to T(\nu_X)$ ← Thom space = p の mapping cone

s.t. $\boxed{[X] = U_{\nu_X} \cap \rho_{X*}[S^{m+k}]}$

U_{ν_X}: ν_X の Thom class

Theorem (Spivak)

For $\forall$ Poincaré cpx X, $\exists$ Spivak normal str. (ν_X, ρ_X)

構成法

Step 1 必要なら X をホモトピー同値な simplicial cpx で置き換えて S^{m+k} $(k \gg 1)$ に部分多面体としてうめこむ: $X \hookrightarrow S^{m+k}$

Step 2 Closed regular nbd $(Y, \partial Y)$ を取る

- Y は ∂ 付き組み合わせ多様体
- X は Y の変位レトラクト

このとき

Claim $\underbrace{\partial Y \hookrightarrow Y}_{\nu_X}$ の mapping fiber は homotopy S^{k-1} ($Y \simeq X$)

- $\rho_X : S^{m+k} \to S^{m+k} / \overline{S^{m+k} \setminus Y} = Y / \partial Y$

- **一意性**

$(\nu, \rho), (\nu', \rho')$: X の 2 つの Spivak normal str. とすると

これらは次の仕で stable に同値 $\left\{ \begin{array}{l} \cdot\ \exists c : \nu \xrightarrow{\simeq} \nu' \text{ (stable fiber h.e.)} \\ \cdot\ T(c)_*[\rho] = [\rho'] \in \pi^S_{m+k}(T(\nu')) \end{array} \right.$

- **ホモトピー不変性**

X, X': 2 つの Poincaré cpx

$f : X \xrightarrow{\simeq} X'$ ホモトピー同値

$\Rightarrow (\nu_X, \rho_X) \underset{\text{stable}}{\simeq} f^*(\nu_{X'}, \rho_{X'})$

分類空間

$G_k := \{ S^{k-1}$ から S^{k-1} へのホモトピー同値写像全体のなすモノイド $\}$

← cpt open topology を入れる

$\rightsquigarrow$

BG_k : 分類空間

$$\left\{ \begin{matrix} (k-1)\text{-spherical fibration の} \\ \text{fiber homotopy class 全体} \end{matrix} \right\} \cong [X, BG_k]$$

- $O_k \to G_k \to G_k/O_k \to BO_k \xrightarrow{J} BG_k$ fibration の sequence

- X : space. もし

$$\begin{tikzcd} & X \arrow[dl, "\tilde{f}"'] \arrow[dr, "f"] \arrow[d, phantom, "\circlearrowleft"] & \\ BO_k \arrow[rr, "J"'] & & BG_k \end{tikzcd}$$

が与えられたとすると

$f \longleftrightarrow (k-1)$-spherical fibration α

$\tilde{f} \longleftrightarrow \alpha$ のベクトル束への reduction

@ 以後 stable なところでものを考えるので

$G := \lim_k G_k$, $O := \lim_k O_k$

$O \to G \to G/O \to BO \xrightarrow{J} BG$

- M^m : smooth manifold, $M^m \underset{\text{embedding}}{\hookrightarrow} S^{m+k}$ $(k \gg 1)$

Stable normal bundle

$\nu_M : M \to BO$ は diffeo. invariant.

一方 $\underbrace{J\nu_M}$ $: M \xrightarrow{\nu_M} BO \xrightarrow{J} BG$ は homotopy invariant.

└ M の Spivak normal str.

- X^m : Poincaré complex

もし $f : M \to X$ がホモトピー同値

$\Rightarrow \exists$ bundle reduction of ν_X

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\nu_X"'] & BO \arrow[d, "J"] \arrow[dl, phantom, "\circlearrowleft"] \\ & BG \end{tikzcd}$$

s.t. $\nu_M \cong f^* \eta$

Def

• (normal map) X: CW cpx, $\eta: X \to BO$ stable bdl

M: smooth mfd, $\nu_M: M \to BO$ stable normal bdl

$(f, b): (M, \nu_M) \longrightarrow (X, \eta)$ が normal map

$\underset{\text{def}}{\Leftrightarrow}$ $f: M \to X$ map, $b: \nu_M \overset{\cong}{\Rightarrow} f^*\eta$

• (normal bordism)

$((F, B); (f, b), (f', b')) : (W; M, M') \to X \times (I; \{0\}, \{1\})$

normal map

Prop. X: finite CW cpx, $\eta_k: X \to BO_k$ given $\rightsquigarrow \eta: X \to BO_k \to BO$

$\left\{ \begin{array}{l} (f,b): (M^m, \nu_M) \to (X, \eta) \\ \text{normal maps} \end{array} \right\} \Big/ \text{normal bordism} \overset{1:1}{\longleftrightarrow} \pi^S_{m+k}(T(\eta)) = \lim_j \pi_{m+k+j}(\Sigma^j T(\eta_k))$

(∵) Pontrjagin-Thom 構成

$[\rho] \in \pi_{m+k}(T(\eta_k))$, $\rho: S^{m+k} \longrightarrow T(\eta_k)$

$$\begin{tikzcd} T(\eta_k) \arrow[d] \\ X \arrow[u, bend right, "s_0 : \text{zero section}"'] \end{tikzcd}$$

もし $\rho \pitchfork s_0 \Rightarrow M := \rho^{-1}(s_0(X))$ smooth mfd

$\rightsquigarrow (f, b): (M, \nu_M) \longrightarrow (X, \eta_k)$ //

さて

X^m: Poincaré cpx とする

(ν_X, ρ_X): Spivak normal structure $\nu_X: X \to BG_k$, $\rho_X: S^{m+k} \to T(\nu_X)$

• ν_X のある bundle reduction $\eta: X \to BO_k$ が given とする

$M := \rho_X^{-1}(s_0(X)) \subset S^{m+k}$

transversality を仮定

$\Rightarrow [X] = U_\eta \cap \rho_*[S^{m+k}]$

$= \rho_*(U_{\nu_M} \cap [T(\nu_M)]) = f_*[M]$

◎ この状況では normal map (f, b) は $\deg = 1$

Def X^m: Poincaré cpx とする

(i) Normal invariant (η, ρ)

$\eta : X \to BO_k \ (k \gg 1)$ ← つまり Spivak normal str. の bundle reduction のこと。

$\rho : S^{m+k} \to T(\eta)$

s.t. $U_\eta \frown \rho_*[S^{m+k}] = [X]$

(ii) $(\eta, \rho) \sim (\eta', \rho')$

$\underset{\text{def}}{\Longleftrightarrow} \ \exists c : \eta \underset{\text{Stable bdl. iso.}}{\xrightarrow{\cong}} \eta'$ s.t. $[\rho'] = T(c)_*[\rho] \in \pi^s_{m+k}(T(\eta'))$

(iii) Normal structure set

$$N(X) := \{(\eta, \rho)\} / \sim$$

Theorem (Browder - Novikov)

⓪ $N(X) \neq \phi \Leftrightarrow \nu_X : X \to BG$ が bundle reduction をもつ

① $N(X) \xleftrightarrow{1:1} \left\{ \begin{array}{l} (f, b) : (M, \nu_M) \to (X, \eta) \ \begin{array}{l}\deg = 1 \\ \text{normal}\end{array} \\ \eta : \nu_X \text{の bdl reduction} \end{array} \right\} \Big/ \text{normal bordism}$

② $N(X) \neq \phi$ のとき基点 $x_0 \in N(X)$ を fix すると

$$N(X) \xleftrightarrow{1:1} [X, G/O]$$

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① Pontrjagin-Thom construction

② (ν_X, ρ_X): Spivak normal str. of X

$[(\eta, \rho)] \in N(X)$

Note ρ の stable homotopy class は ρ_X から一意に決まる

結局

$N(X) \cong \{\nu_X \text{の bdl reduction } \eta \text{ の同型類}\}$

$\underset{\text{基点}}{\cong} [X, G/O]$ //