「低次元幾何学 と 無限次元幾何学」 '07 Sept. 12

Generalized rational blow-down and Euclidean Algorithm

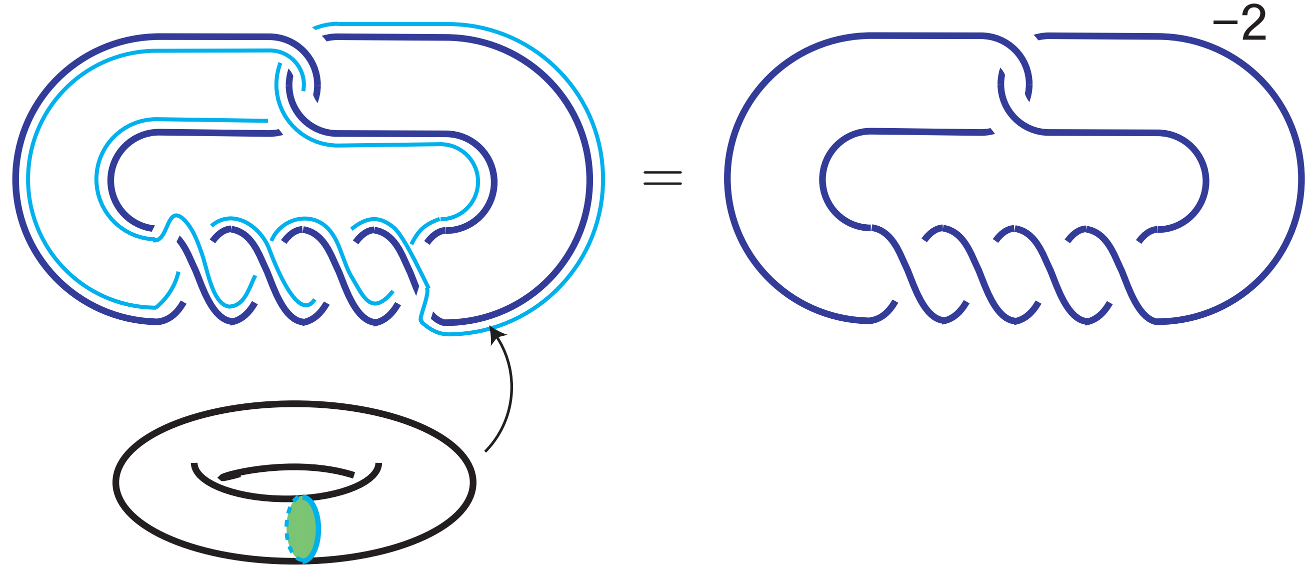
山田　裕一　(電気通信大学)

今日の標語.

Lens space surgery 族 に潜む特異点解消 –Euclidean Algorithm–

§0. Dehn surgery, Kirby Calculus

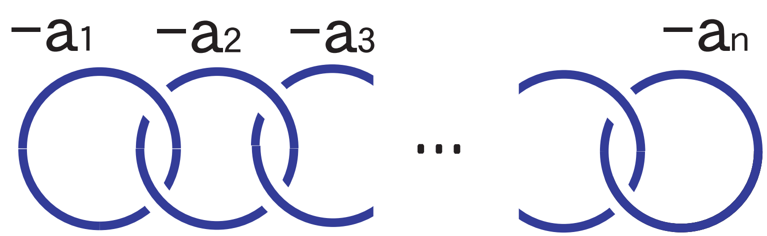
Dehn surgery coefficient = Framing = a parallel curve, or the linking number. *Remove and Reglue* Solid torus along each component such as "the meridian comes to the parallel"



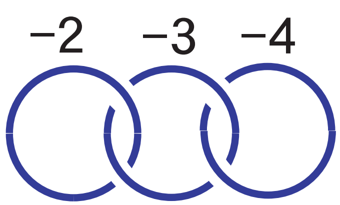
Thm. [Lickorish '62]
Any closed connected oriented 3-manifold is obtained by a framed link L in S^3. Notation: $M(L)$, or $(K; p)$.

Framed Links for Lens $L(p,q)$

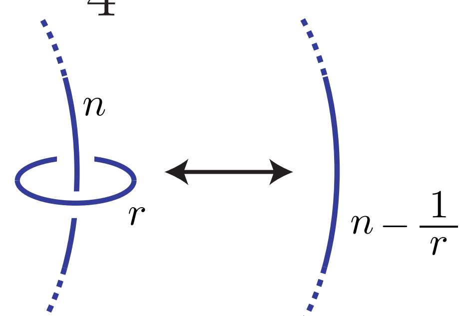
$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$



$L(18,11)$ $(= L(18,5))$ $\qquad \frac{18}{11} = 2 - \cfrac{1}{3 - \cfrac{1}{4}}, \quad \frac{18}{5} = 4 - \cfrac{1}{3 - \cfrac{1}{2}}$



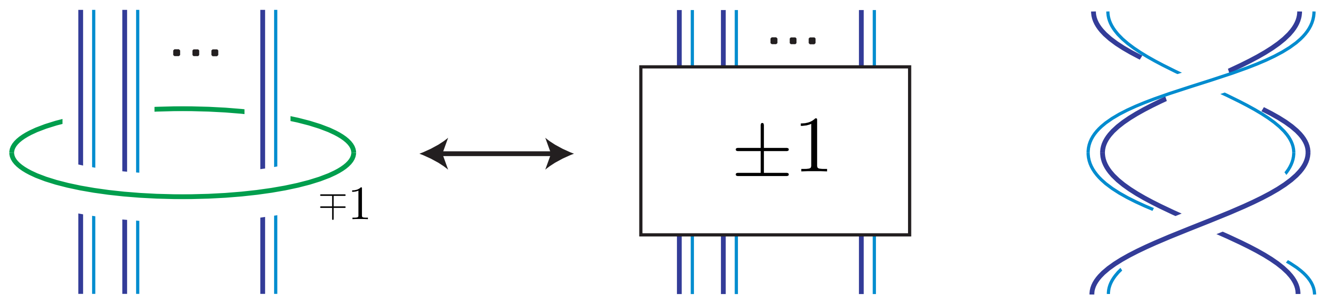
For $n \in \mathbf{Z}, r \in \mathbf{Q}$



Thm. Kirby Calclus ([Fenn-Rourke] ver.)

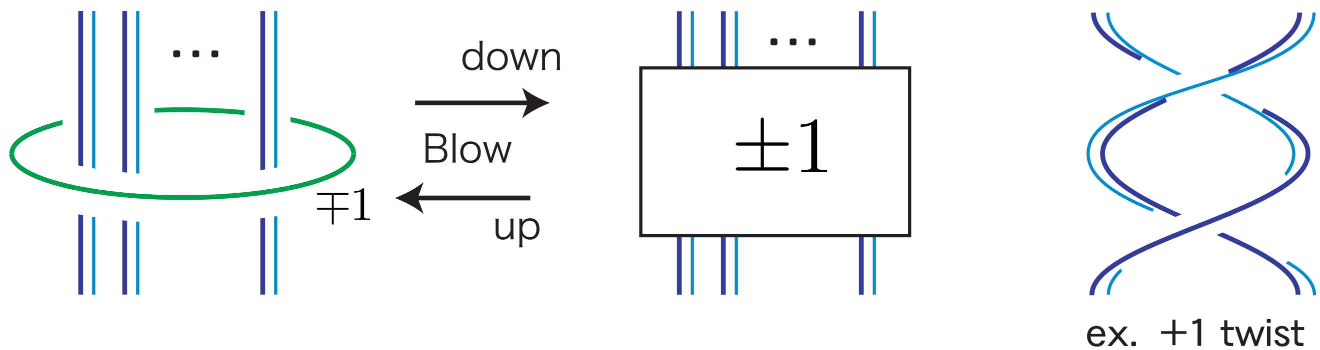
Framed links L_1, L_2 are moved to each other as below and isotopy,

$\Leftrightarrow\ M(L_1) \cong M(L_2)$.

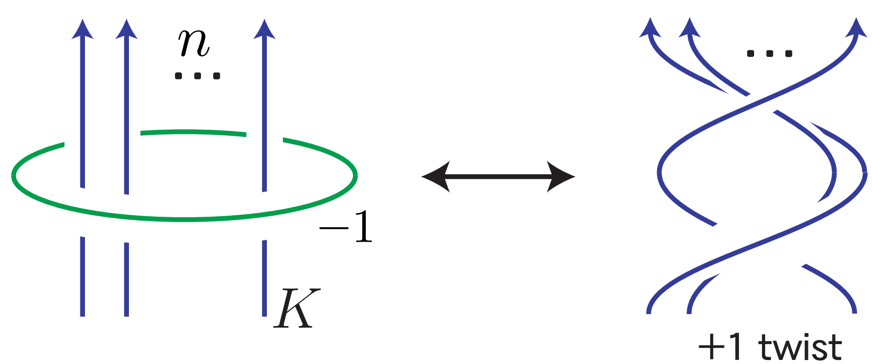


ex. +1 twist

Thm. Kirby Calclus ([Fenn-Rourke] ver.)
Framed links L_1, L_2 are moved to each other as below and isotopy,
$\Leftrightarrow$ $M(L_1) \cong M(L_2)$.

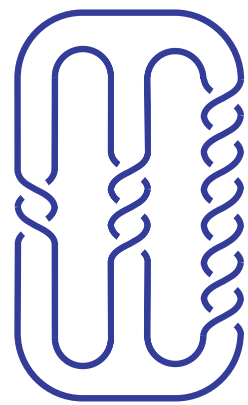


Special case: If the blue is connected (K) and orientations are same,



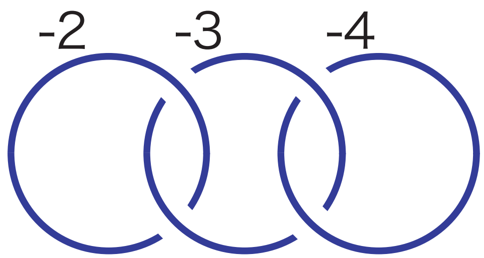
The framing inclease by n^2.

Q. 証明できる



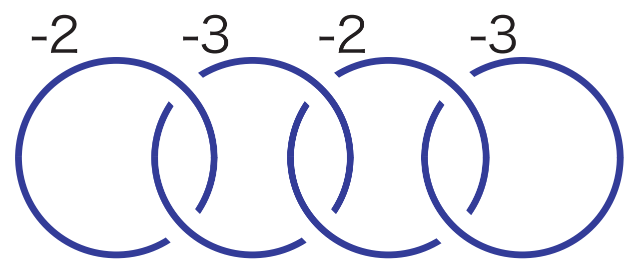
P(-2,3,7)

18-surgery is



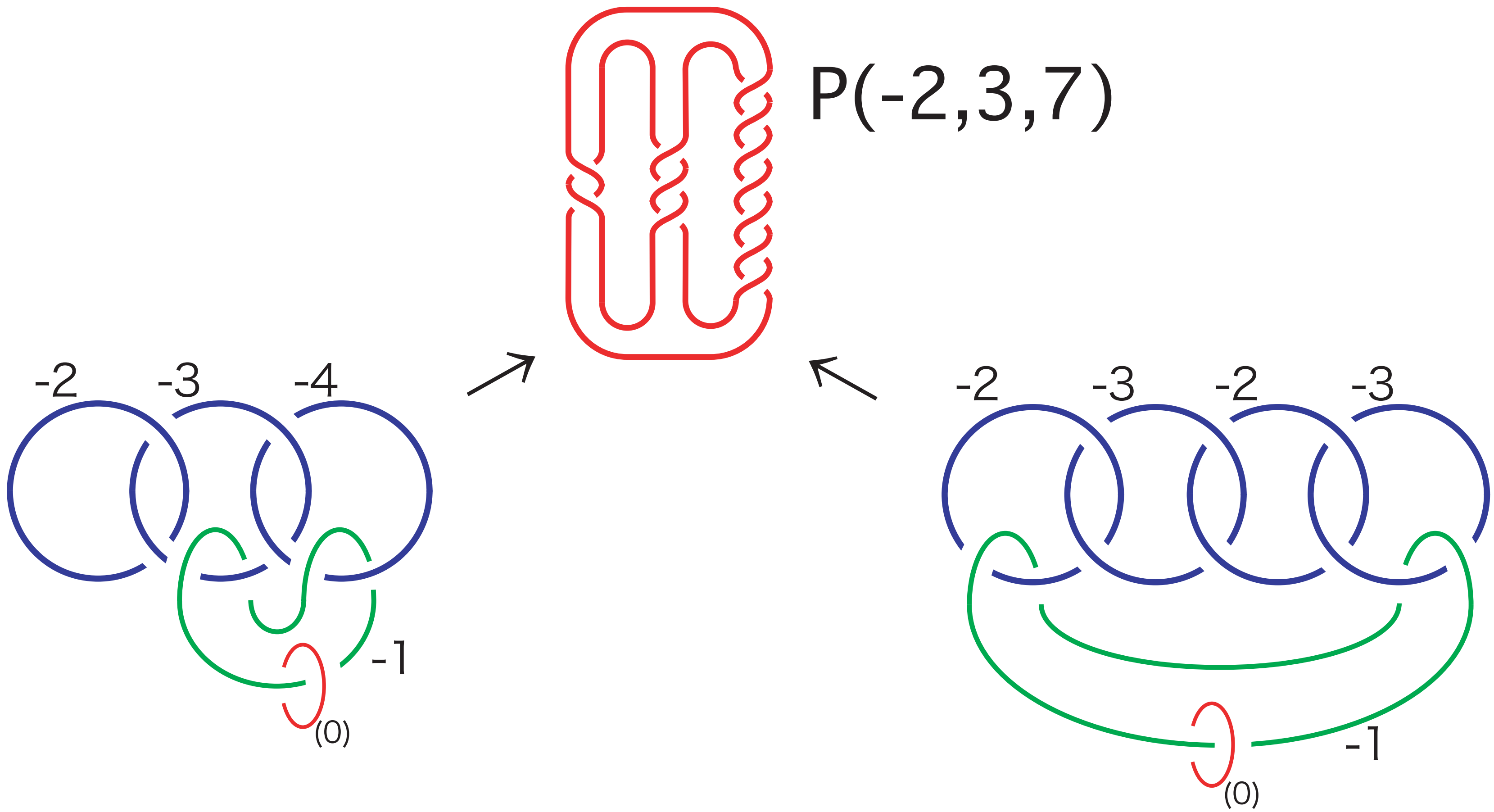
?

19-surgery is



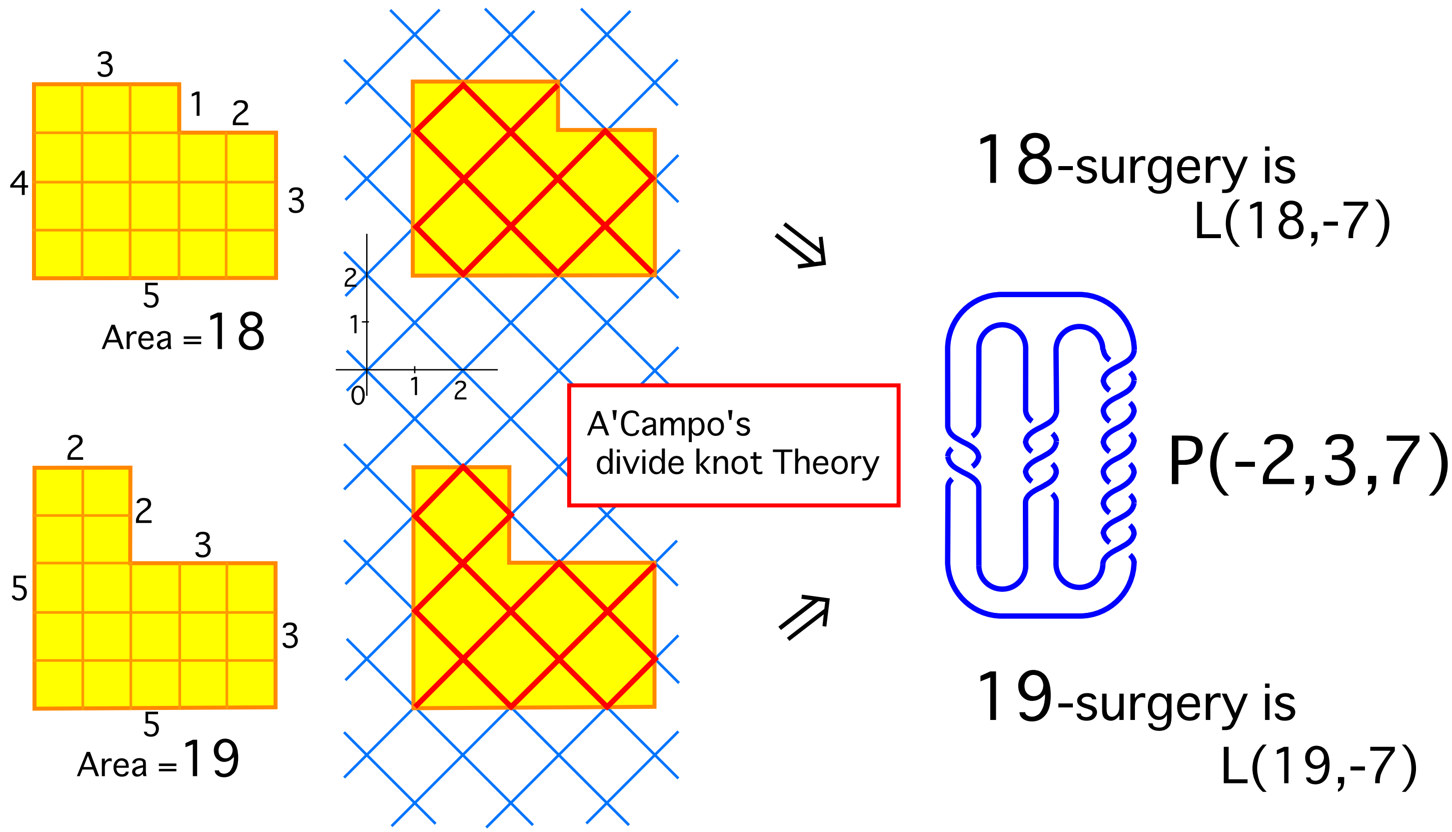
とはいうものの…

Answer. [Fintushel-Stern]([Y])



They are related to Resolution!

§1. My reserach. Lens space surgery.

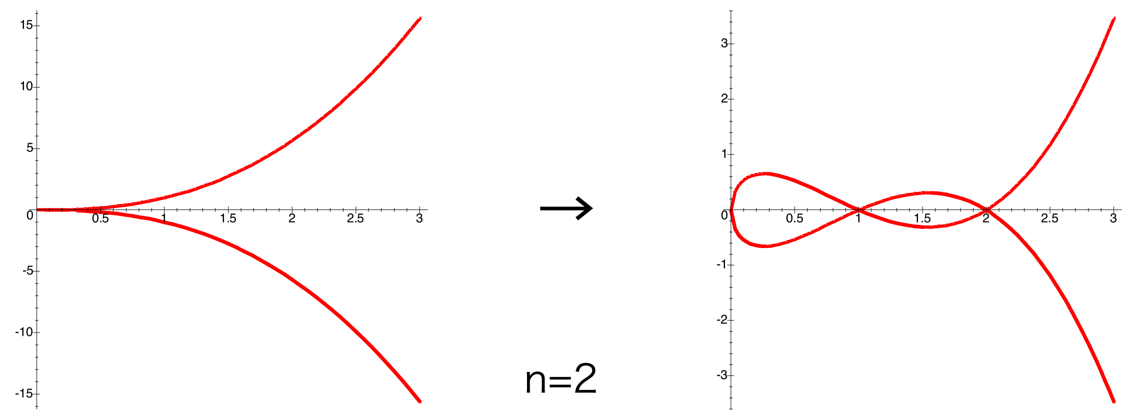


A'Campo's "divide knot theory" comes from *singularity*.

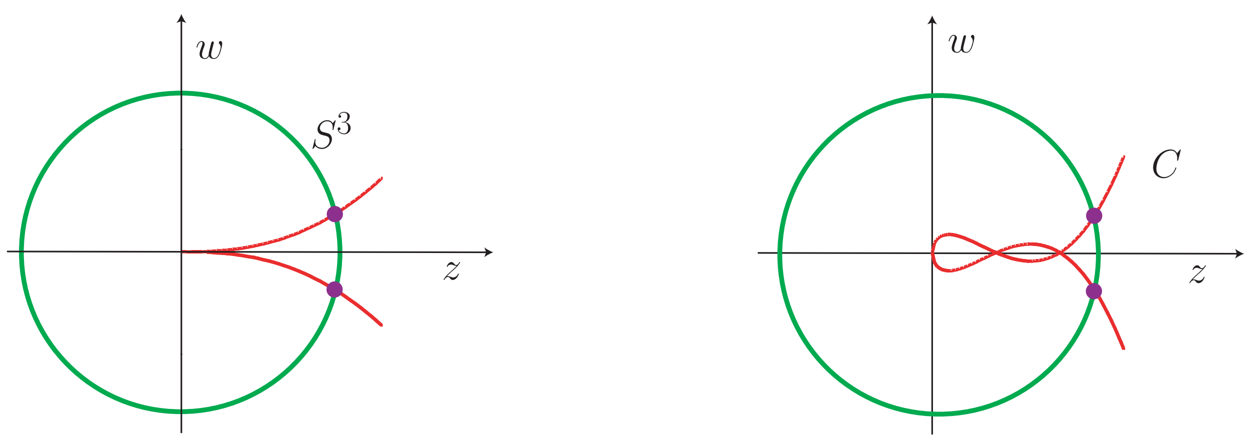
A'Campo's divide knots: Singularity

In $\mathbf{R}^2$, purterb $y^2 = x^{2n+1}$ ("A_{2n}-sing.") and Draw the plane curve:

$$C: \quad y^2 = x(x-\epsilon)^2(x-2\epsilon)^2 \cdots (x-n\epsilon)^2$$



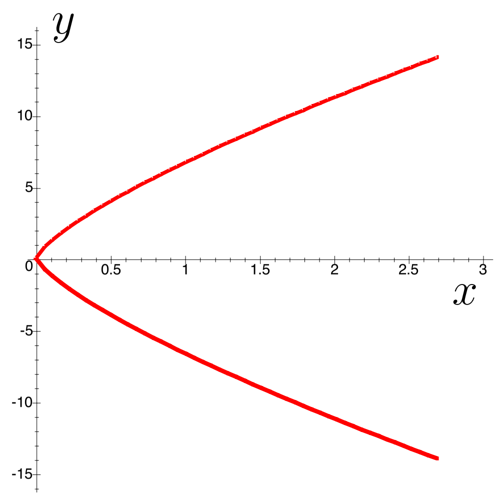
In $\mathbf{C}^2$, $C \cap S^3$ is a knot (or a link). For small ϵ, it is $T(2, 2n+1)$.



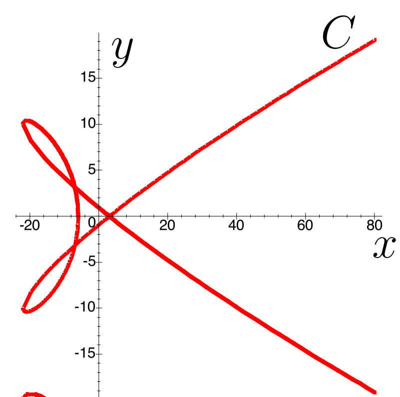
Another example [T.Urabe]

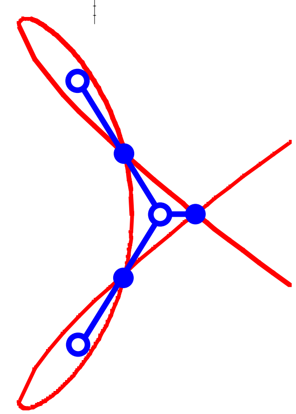
In $\mathbf{R}^2$, purterb $y^4 = x^3$ ("E_6-sing.") and Draw the plane curve:

$$C : \quad (y^2 + \epsilon(6x + 32\epsilon^2))^2 - (x + 7\epsilon^2)^2(x + 22\epsilon^2) = 0$$



$\rightarrow$





The knot is $T(4,3)$. We can see E_6 *Dynkin diagram.*

A'Campo generalized such correspondence

generic Plane Curves P $\Rightarrow$ Links $L(P)$ in S^3

A'Campo's divide knots

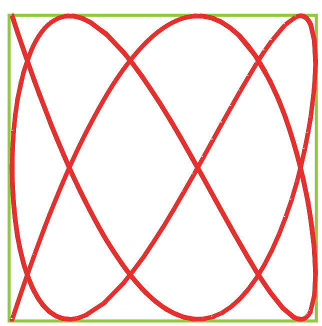
See 平澤(Hirasawa)'s visualized method).

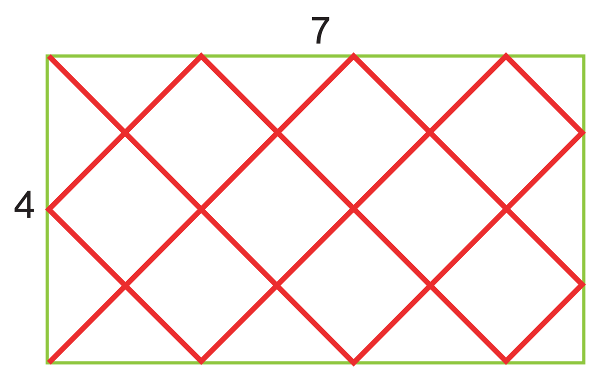
Ex. Torus links [Goda-Hirasawa-Y, (Gusein-Zade, etc.)]

Lissajous curve (p,q) $\sim$ Billiard curve (p,q) $\Rightarrow$ $T(p,q)$

$(\cos p\theta, \cos q\theta)$ PL line with slope ± 1 in a $p \times q$ Rectangle.

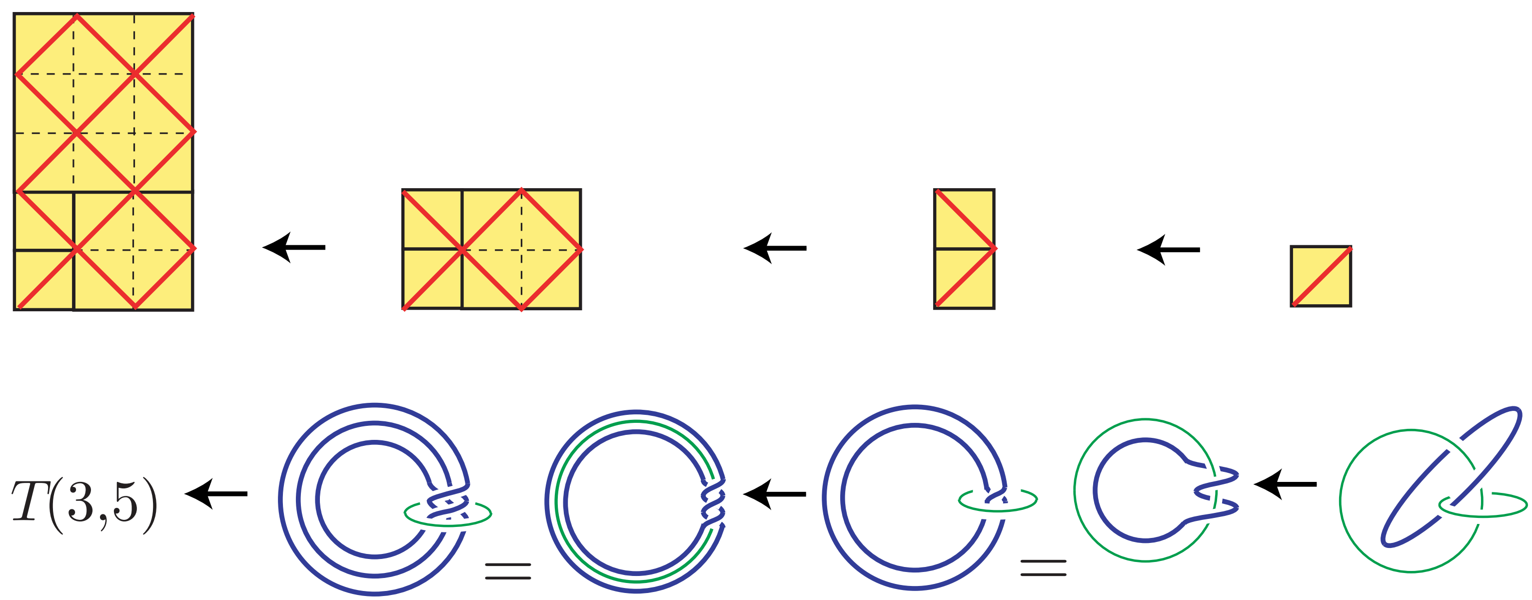
ex. $(p,q) = (7,4)$





This curve has 9 double points, $\dfrac{(p-1)(q-1)}{2}$ in general.

Torus knot is obtained by twisting



related to **Euclidean Algorithm**.

$$(3,5) \quad \to_R \quad (3,2) \quad \to_L \quad (1,2) \quad \to_R \quad (1,1)$$

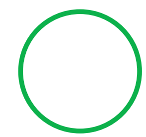
Blow Up

$$\begin{array}{ccc} \mathbf{C}^2 \times \mathbf{C}P^1 & \supset & U := \{((z,w),[s:t]) \,|\, zt = ws\} \\ \downarrow_{pr.1} & & \downarrow \pi := pr.1|_U \\ \mathbf{C}^2 & & (z,w) \end{array}$$

Note

(1) $\pi^{-1}(z,w) = ((z,w),[z:w])$ for $(z,w) \neq (0,0)$

(2) $\pi^{-1}(0,0) \cong \mathbf{C}P^1 = S^2 \qquad (U \cong \text{punc}\overline{\mathbf{C}P^2})$



-1

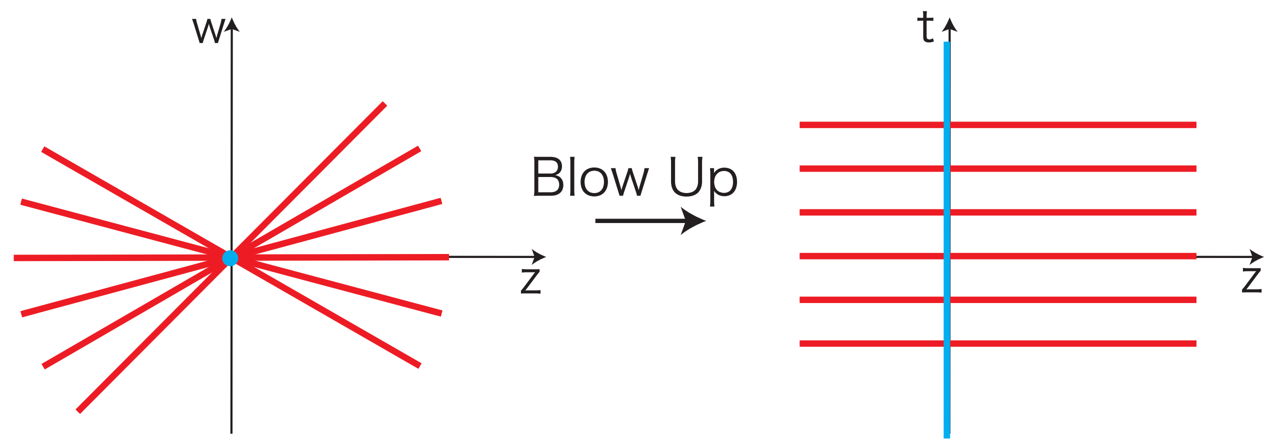
local coordinates of U

$$\begin{array}{ccccc} \mathbf{C}\times\mathbf{C} & \to & U & \leftarrow & \mathbf{C}\times\mathbf{C} \\ (z,t) & \to & ((z,zt),[1:t]) & & \\ & & ((ws,w),[s:1]) & \leftarrow & (w,s) \\ (z,t) & & \to & & (zt,\frac{1}{t}) \end{array}$$

In $\mathbf{C}^2$, $(z,w) = (z,zt) = (ws,w)$.

By $(z, w) = (z, zt)$,

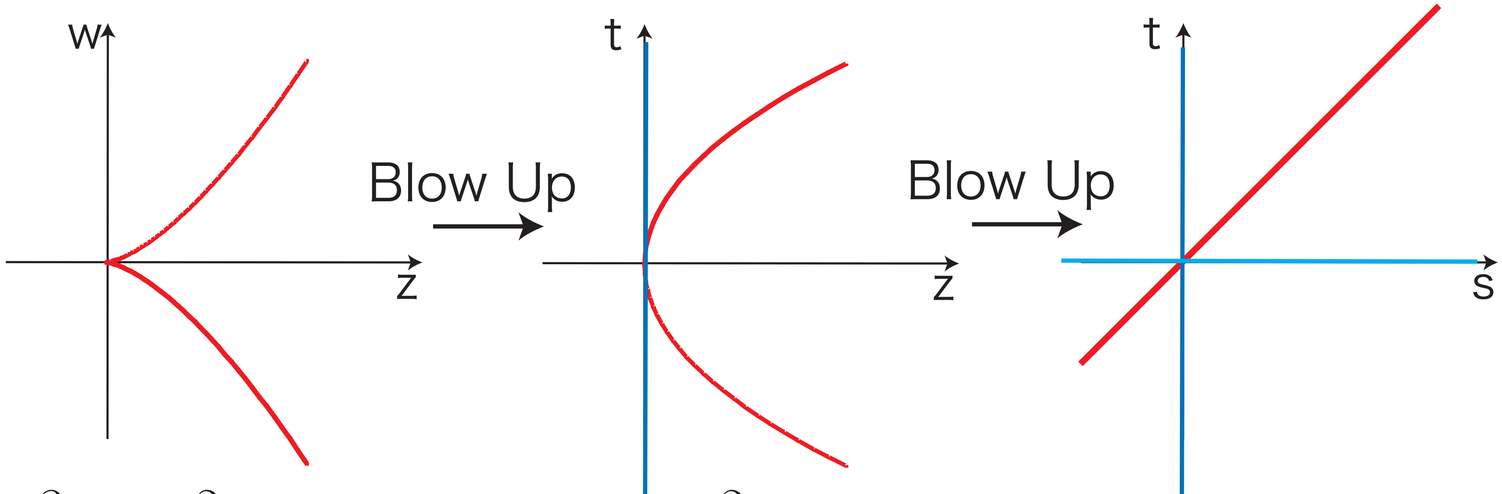
intersecting lines $\cup_i \{w = a_i z\}$ become parallel $\cup_i \{t = a_i\}$



New curve $\{z = 0\}$ (-1-curve $\mathbb{P}^1$) is "exceptional curve".

Singularity at $(0,0)$ of $\{f = 0\}$ in zw-plane, It's lift $\pi^{-1}(f=0) \subset U$ in zt- or ws- plane is "milder"

Ex.1 $T(2,3)$ becomes $T(2,1)$ by Blow up.



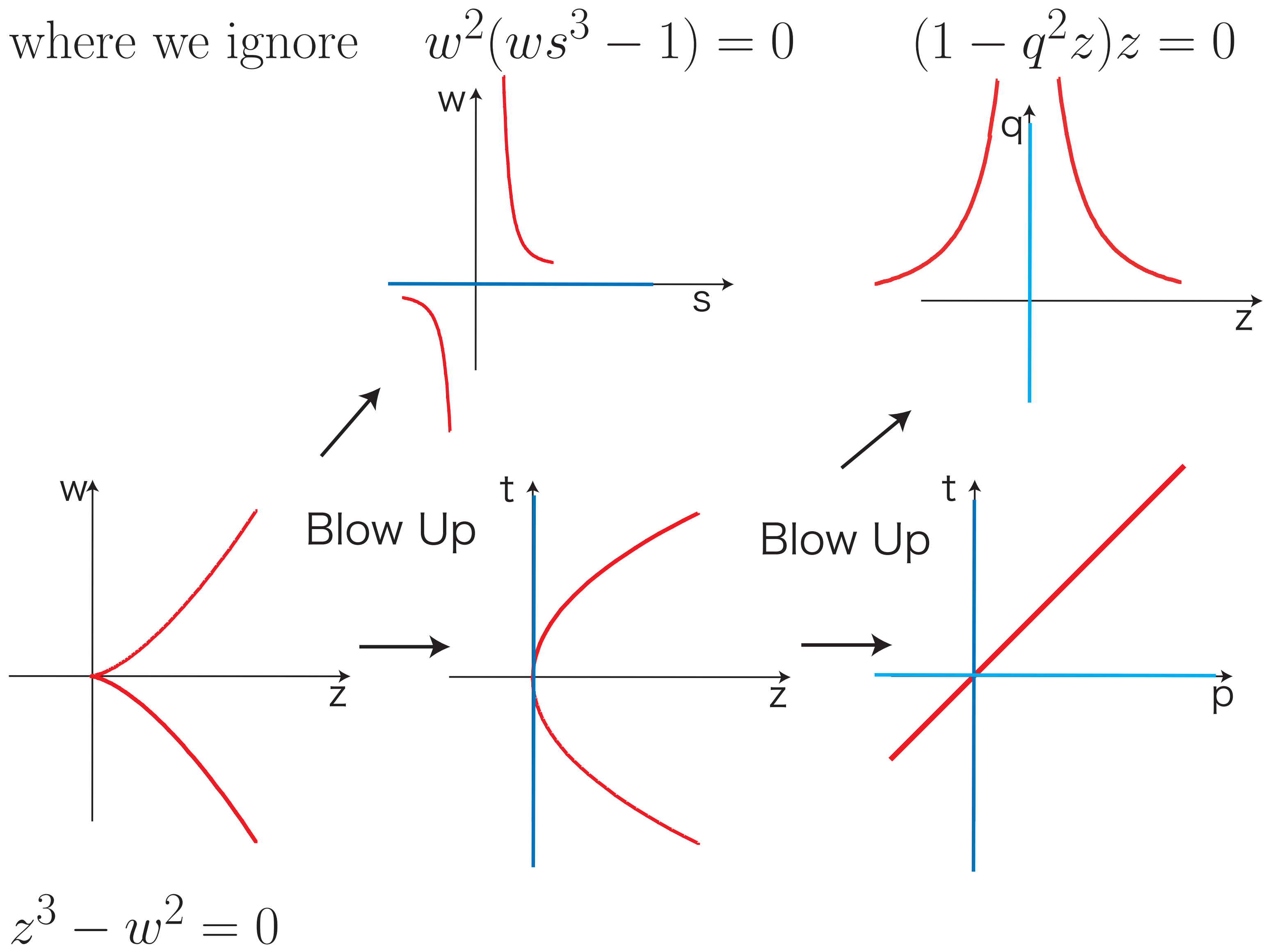
$z^3 - w^2 = 0$ $\quad$ $z - t^2 = 0$ $\quad$ $s - t = 0$

$$z^3 - w^2 = 0 \quad \Rightarrow \quad z^3 - (zt)^2 = 0$$

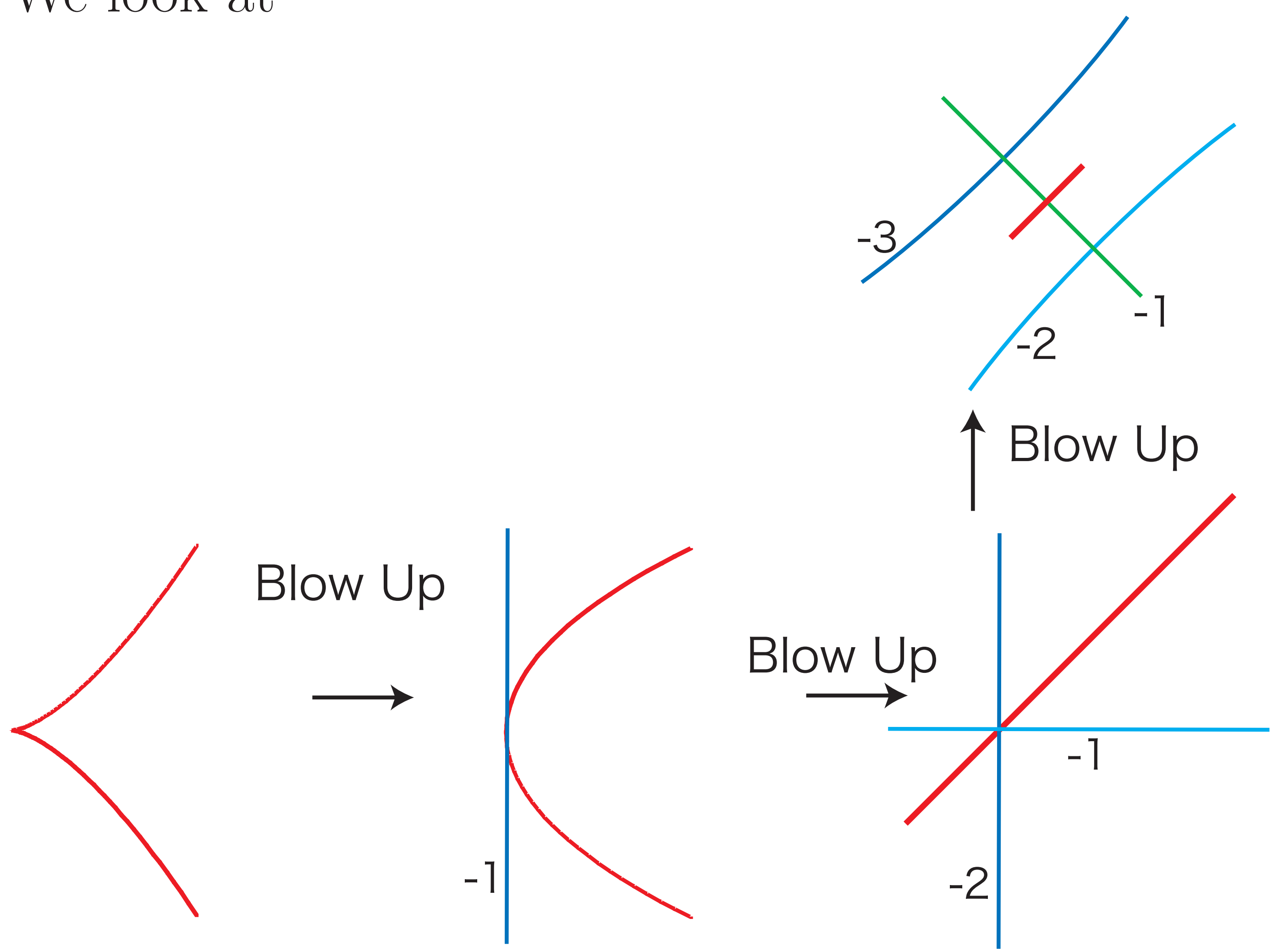
$$z^2\,(z - t^2) = 0. \quad z = 0 \text{ is a new curve.}$$

$$z - t^2 = 0 \quad \Rightarrow \quad ts - t^2 = 0$$

$$t\,(s - t) = 0. \quad t = 0 \text{ is a new curve.}$$


where we ignore
$w^2(ws^3 - 1) = 0$
$(1 - q^2 z)z = 0$
w
s
q
z
w
z
Blow Up
t
z
Blow Up
t
p
$z^3 - w^2 = 0$

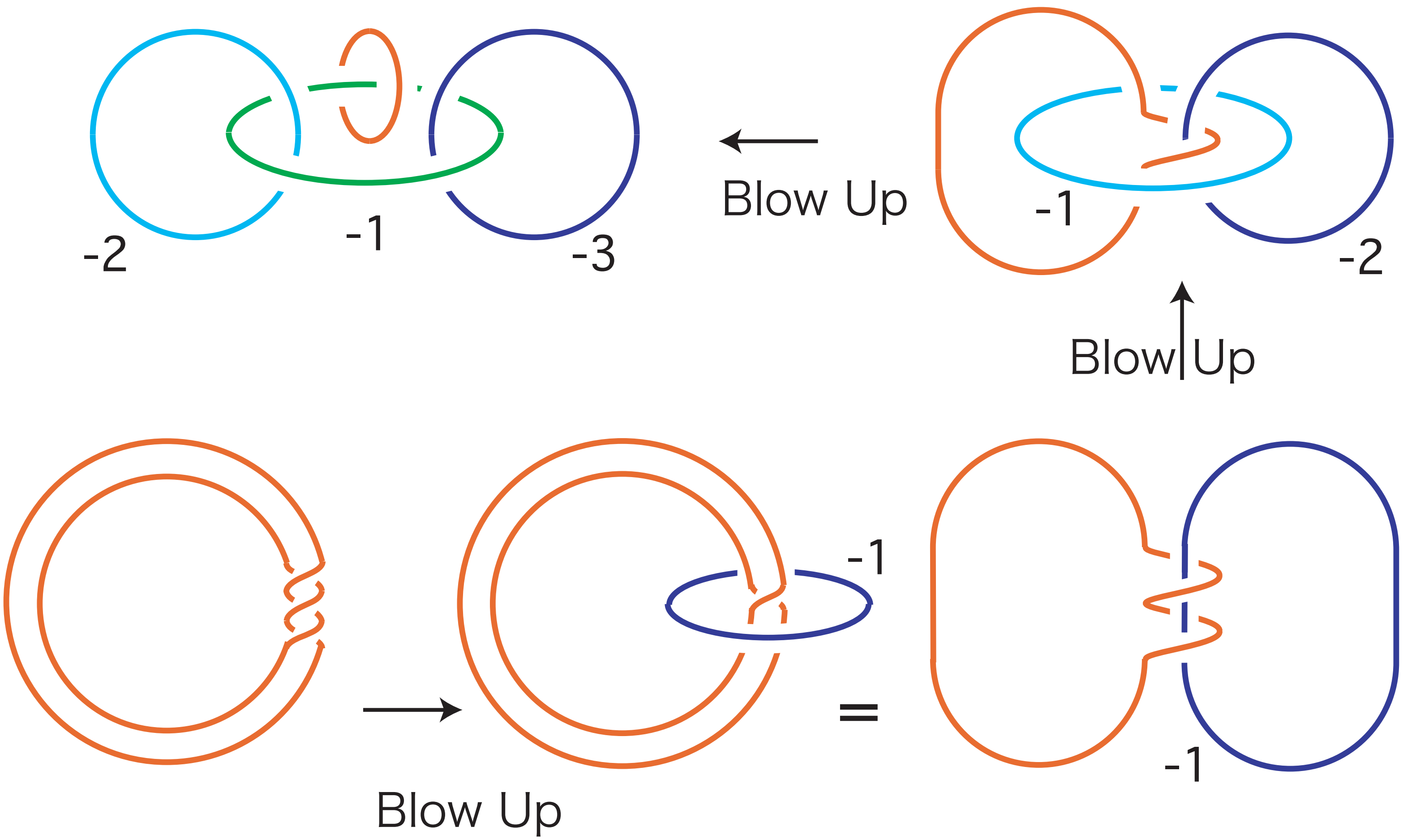
We look at



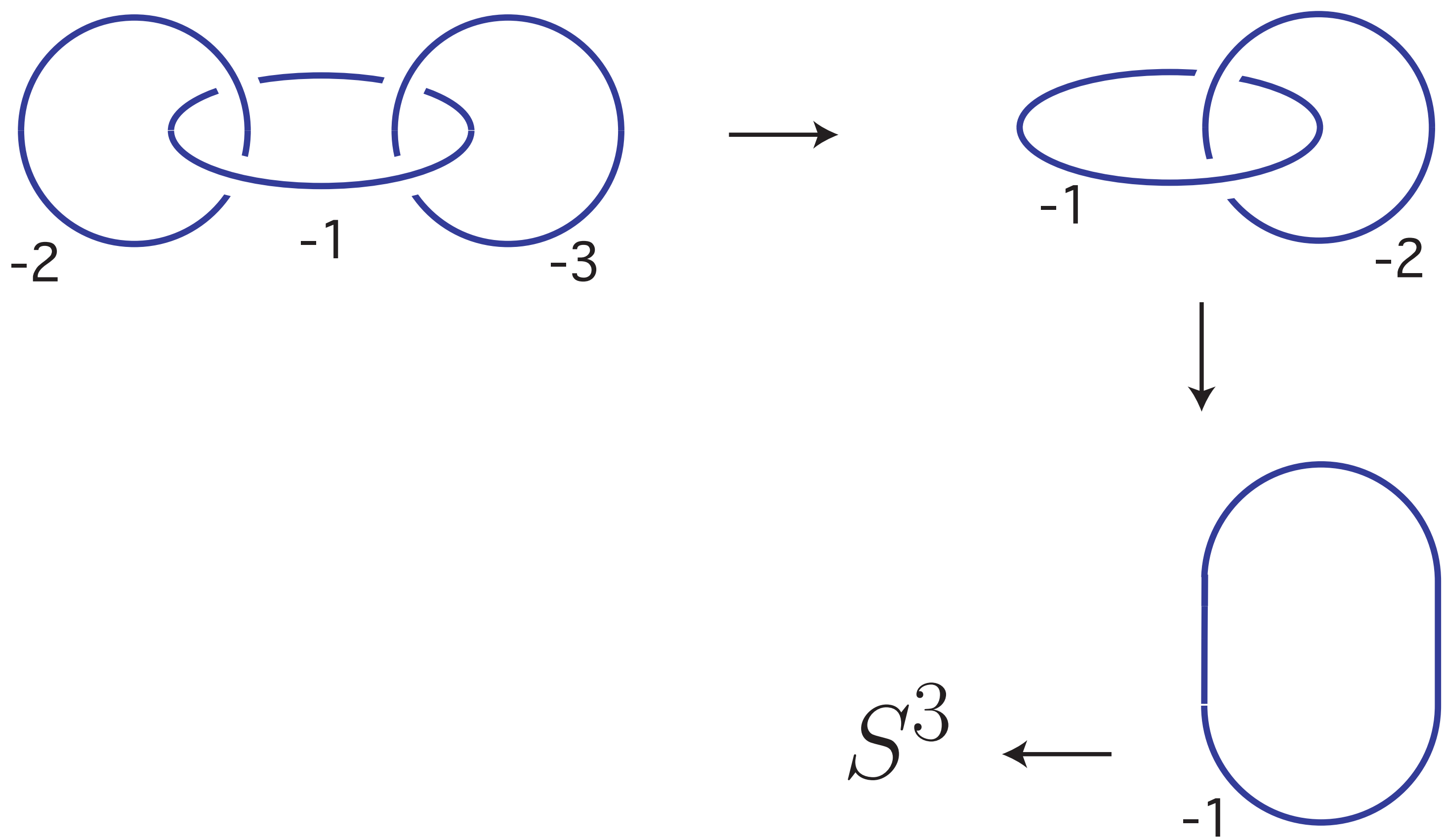
$z^3 - w^2 = 0$

$\Rightarrow$ By Kirby Diagrams,

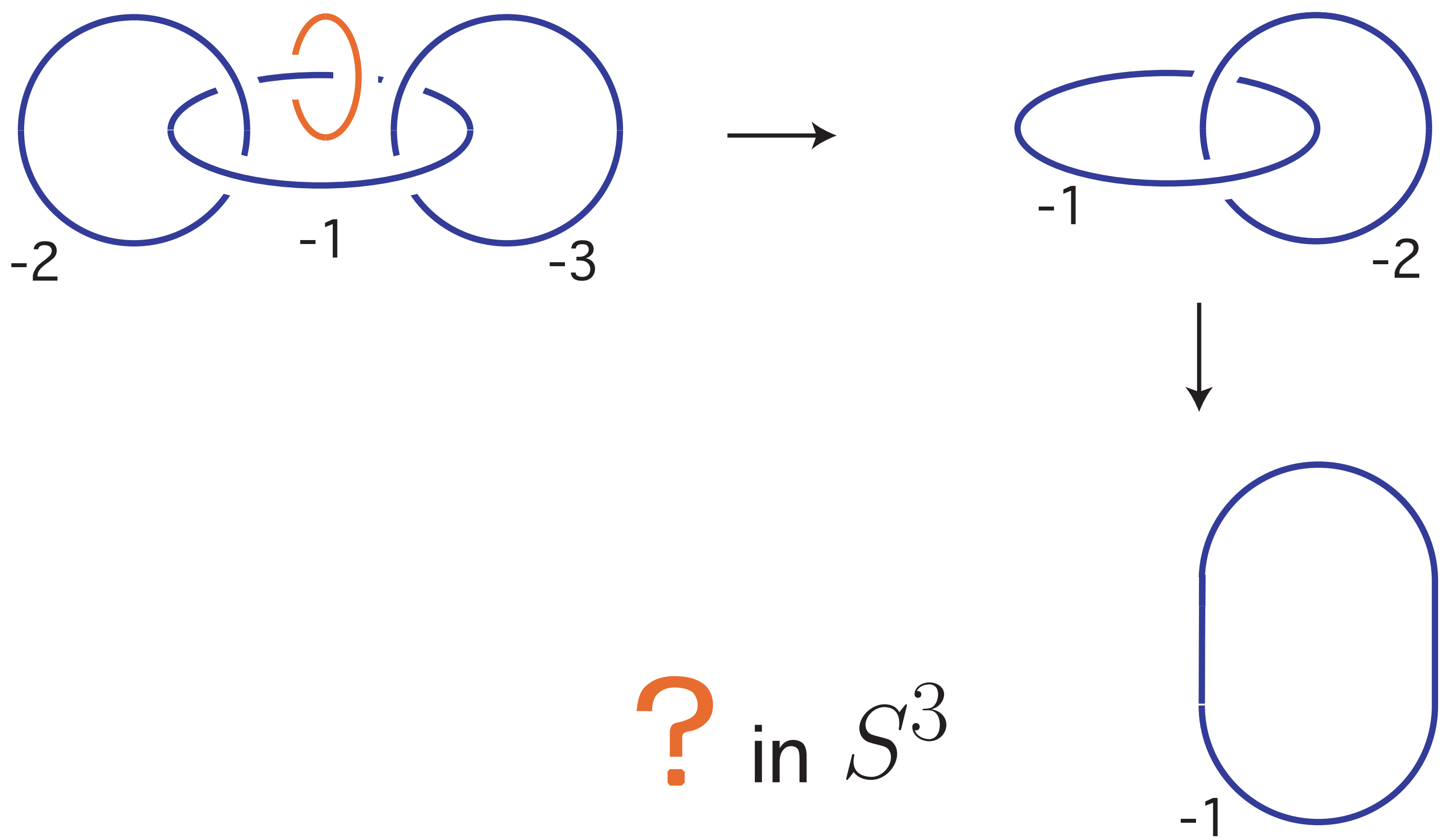
By Kirby Diagrams,



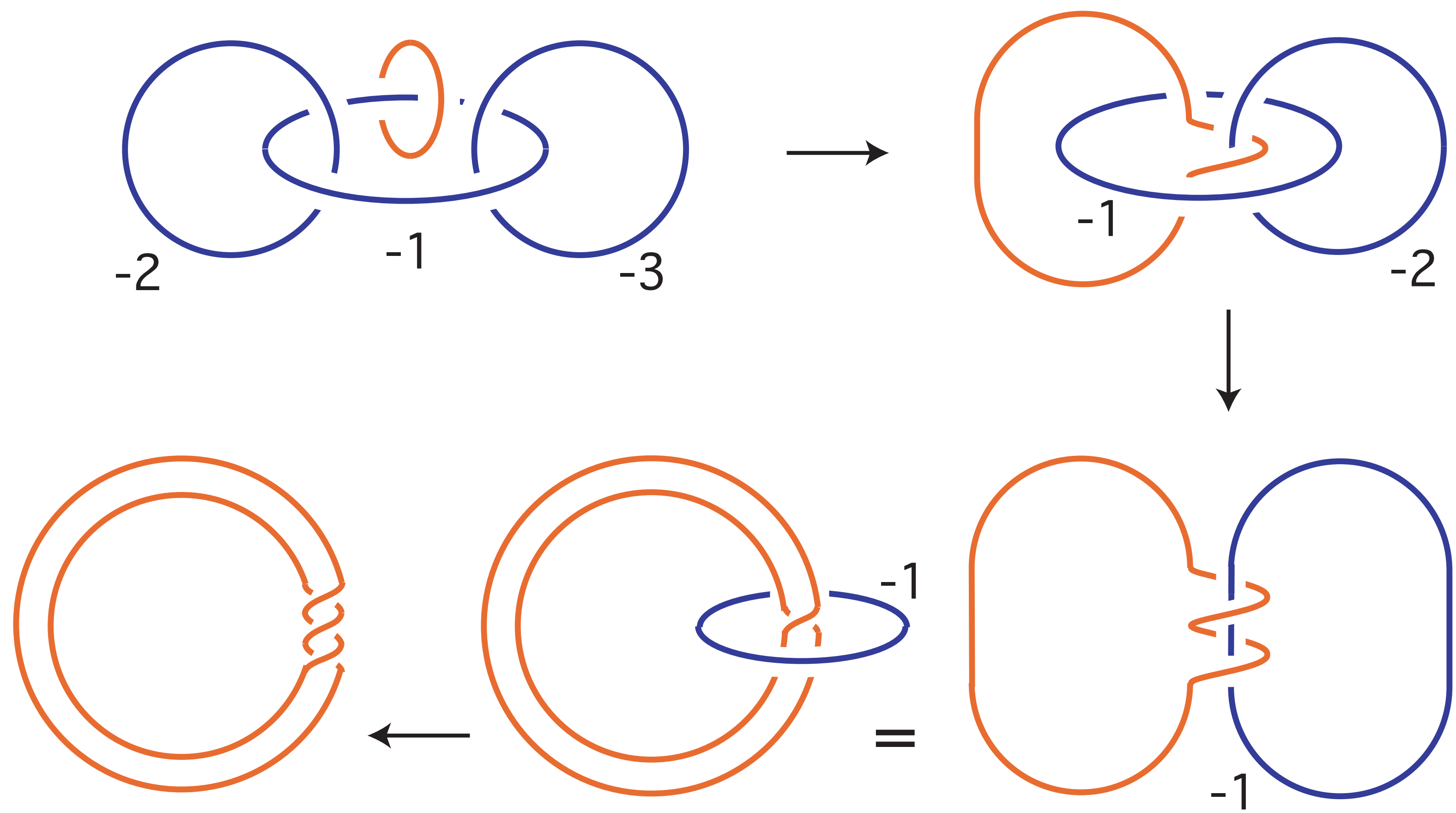
This is $\mathbf{S}^3$



What is this knot?

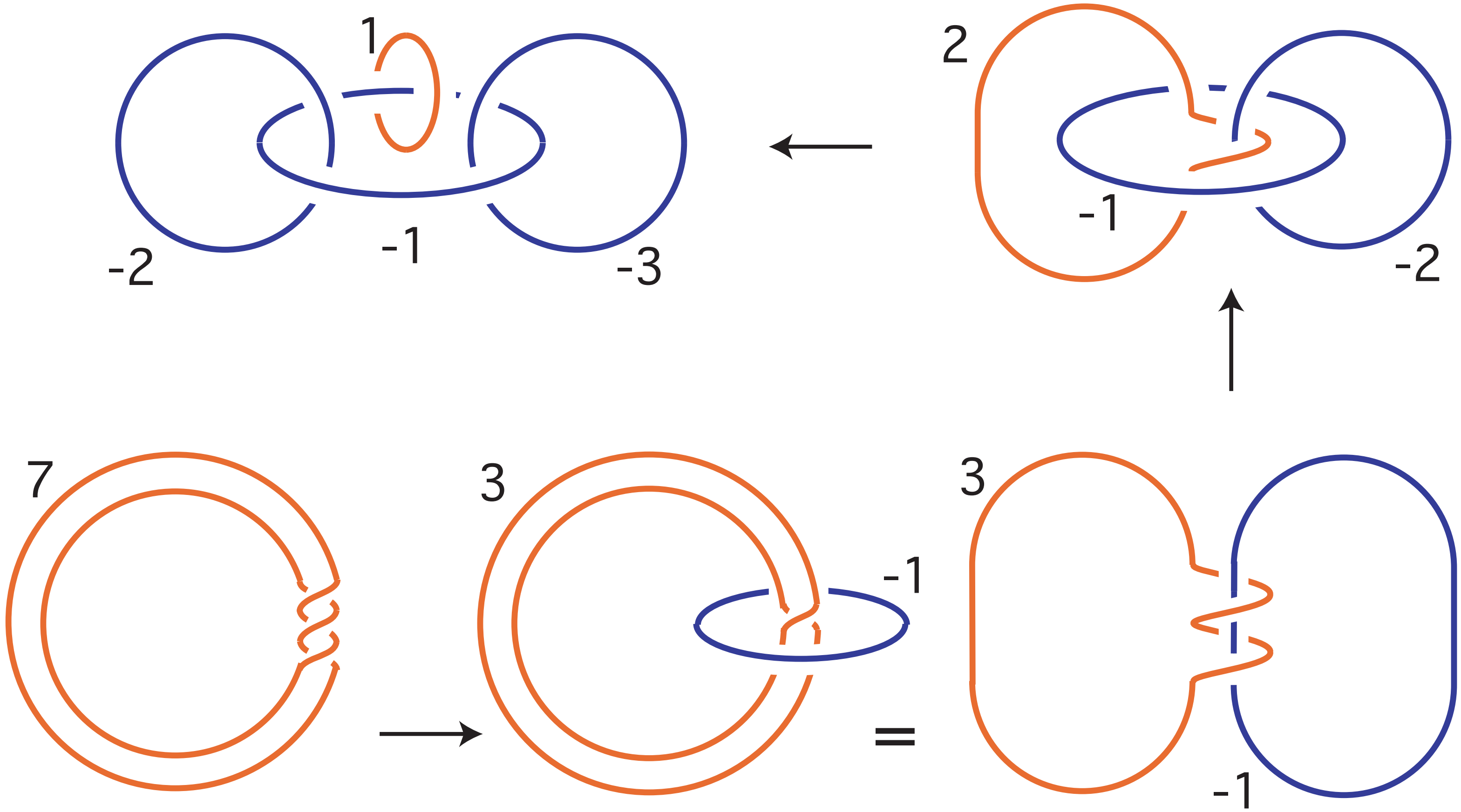


In S^3, the knot is trefoil $T(2,3)$.

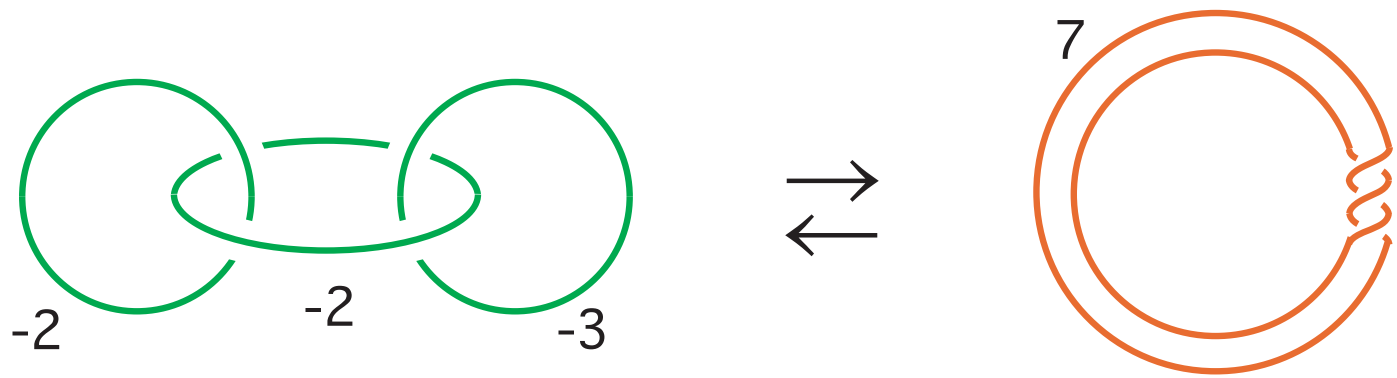


What is 7-surgery along $T(2,3)$?

7-surgery along $T(2,3)$: $(T(2,3);7)$ is



Answer: $(T(2,3);7) = L(7,5)$. $[2,2,3] = 2 - /(2 - 1/3) = \frac{7}{5}$.

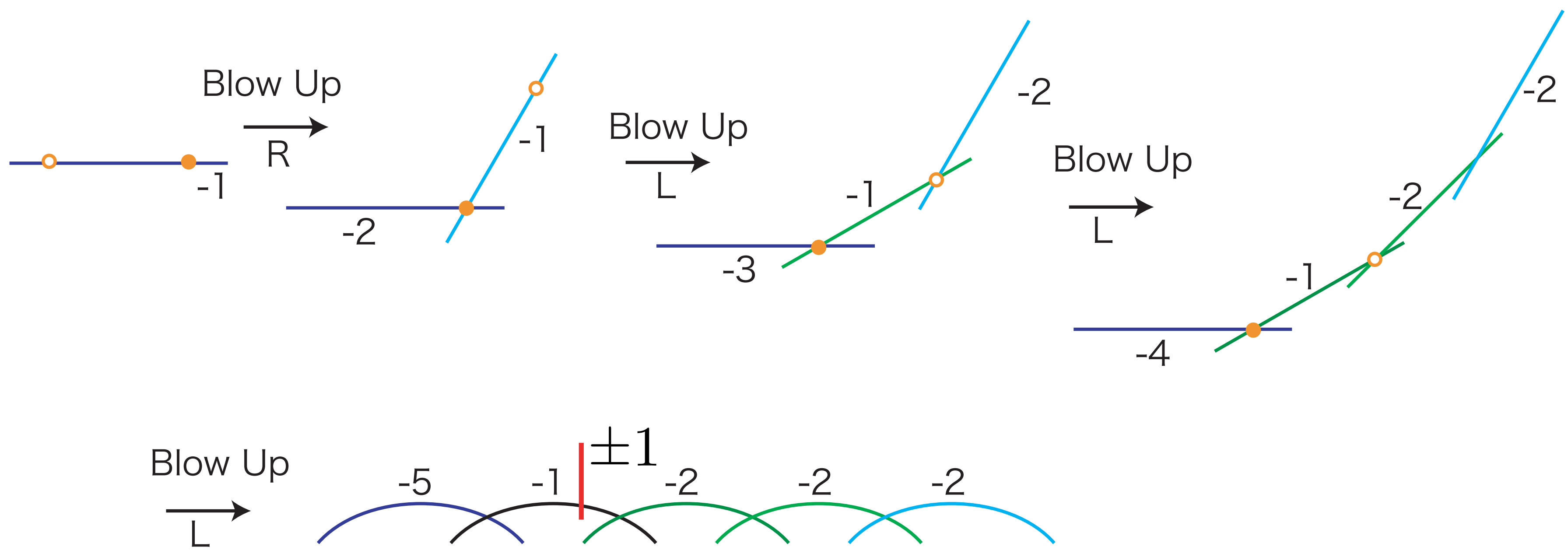


In general, *lens surgery along* $T(p,q)$
is related to Resolution of singularity (= Blow up) of $z^q - w^p = 0$.
(**Euclid Algorithm**)

$$(4,5) \to_R (4,1) \to_L (3,1) \to_L (2,1) \to_L (1,1)$$

源 Euclidean Algorithm, Blow-Up and Torus knot

$$(4,5) \;\to_R\; (4,1) \;\to_L\; (3,1) \;\to_L\; (2,1) \;\to_L\; (1,1)$$



$$[5,2,2,2,2] = \frac{21}{5}$$

$(T(4,5);21) = L(21,5)$. Non-trivial knots can yield lens spaces!

Lens surgery. in classical sense

ex.1 ['71 L. Moser] **Torus knots.**

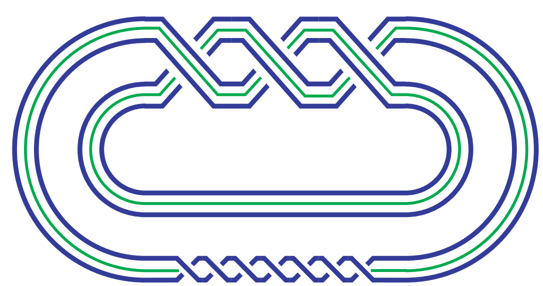
$p = ab \pm 1 \;\Rightarrow\; (T(a,b); p) \cong L(p, -b^2)$.

$K := T(2,3)$, then $(K;7) = L(7,5)$ and $(K;5) = L(5,1)$.

$p = ab \;\Rightarrow\; (T(a,b); p) \cong L(a,-b) \sharp L(b,-a)$.

ex.2 ['77 J. Bailey, D. Rolfsen] **2 Cables of Torus knots**

– Omitted ($\to$ Tange's talk?) –

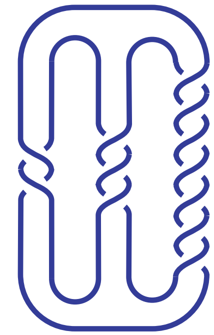


ex.3 ['80 R. Fintushel, R. Stern] Hyperbolic knot!

$K := Pr(-2,3,7)$, then $M(K,19) = -L(19,7)$.

$$M(K,18) = -L(18,7).$$

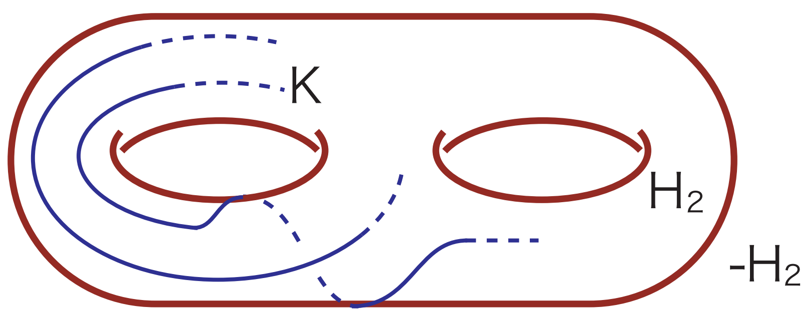
$$\pi_1(\,M(K,17)\,) \text{ is finite, not cyclic.}$$



J. Berge's doubly-primitive knots ['90]

A knot K in the Heegaard surface Σ_2 is *doubly-primitive* iff

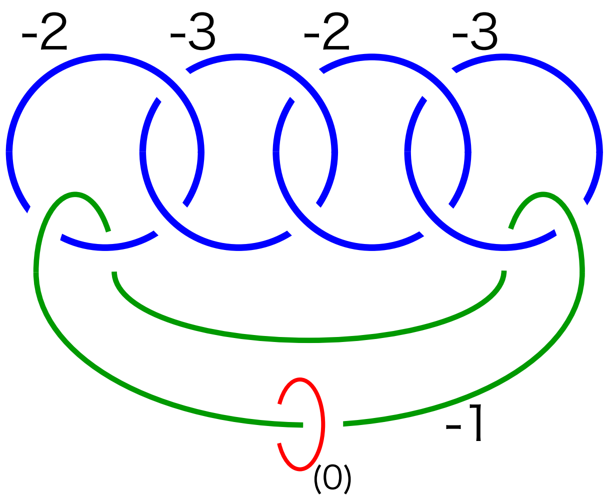
$$K_\sharp \text{ (as in } \pi_1\text{) is a generator in both } \pi_1(H_2) \text{ and } \pi_1(-H_2).$$



He classified and listed them up (or tried to do it). □

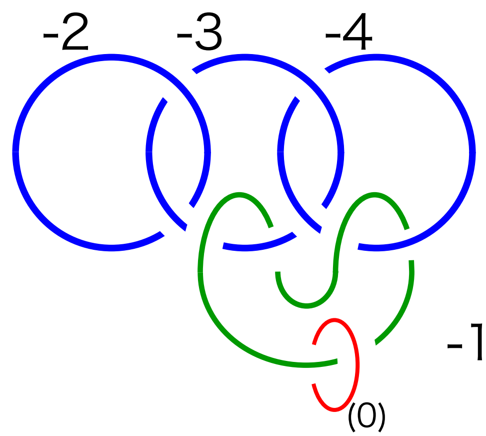
These lens surgeries are proved by Framed links [Y]

19-surgery $L(19, 8)$



$$[2, 3, 2, 3] = \frac{19}{12}$$

18-surgery $L(18, 11)$



$$[2, 3, 4] = \frac{18}{11}$$

(1) Blue link describes the lens space.

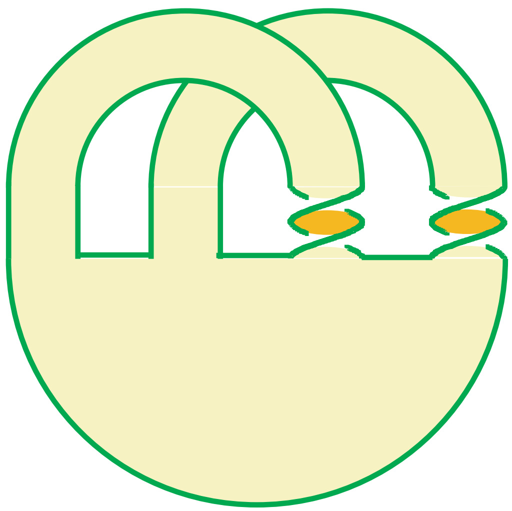
(2) Blue∪green is S^3,

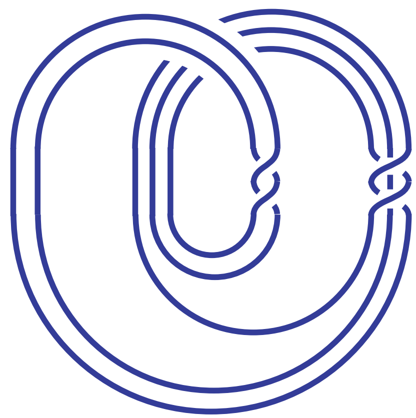
(3) which contains red knot as $P(-2, 3, 7)$ with 18, 19-framed.

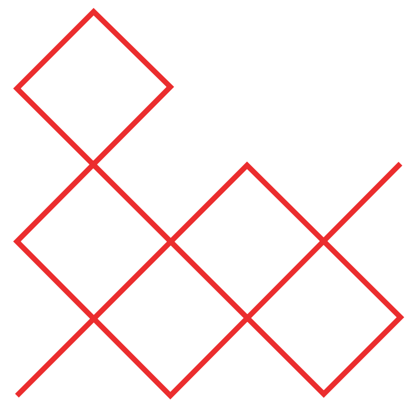
Lens space surgery of TypeVII.

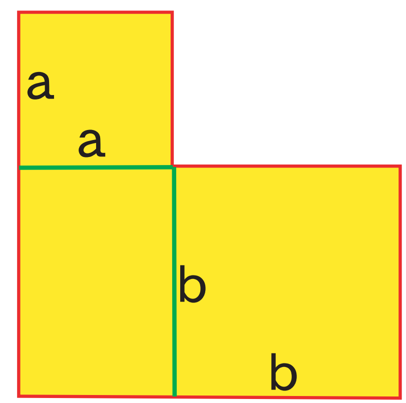
A knot $k^+(a,b)$ in a fiber surface F of Trefoil with p-framing is $L(p,q)$. $\quad$ ($p = a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)

- $(a,b) = (2,3) \Rightarrow P(-2,3,7)$



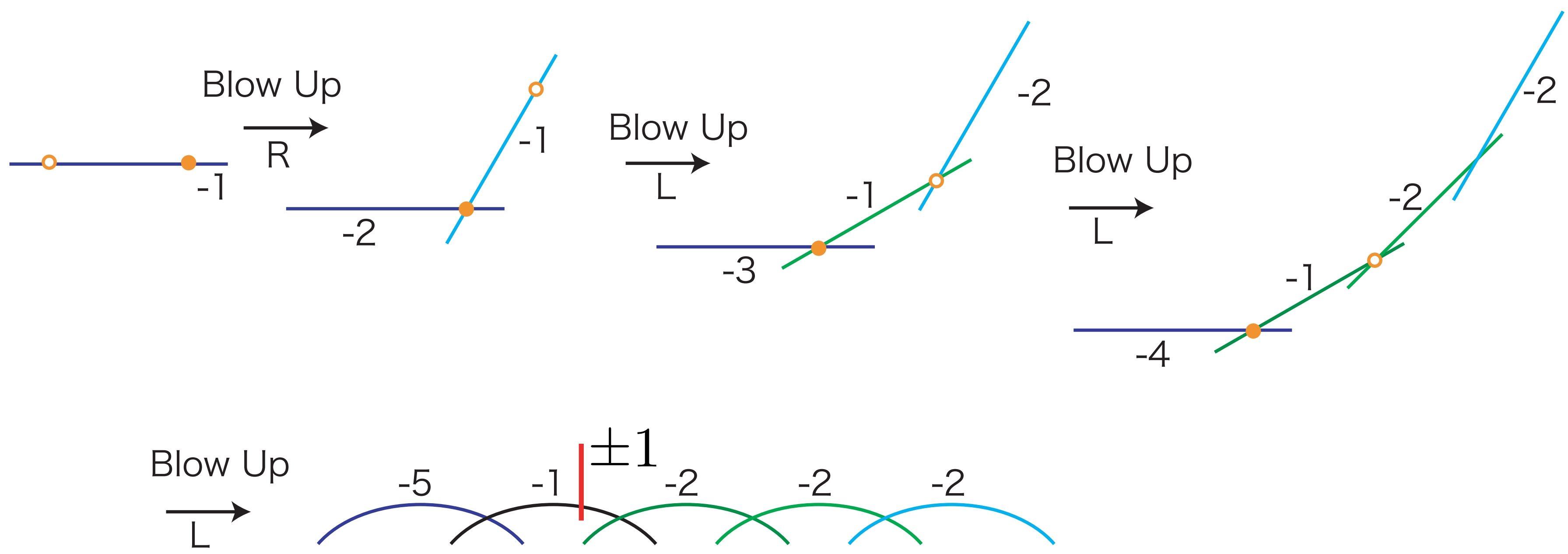






さっきの構成

$$(4,5) \longrightarrow_R (4,1) \longrightarrow_L (3,1) \longrightarrow_L (2,1) \longrightarrow_L (1,1)$$

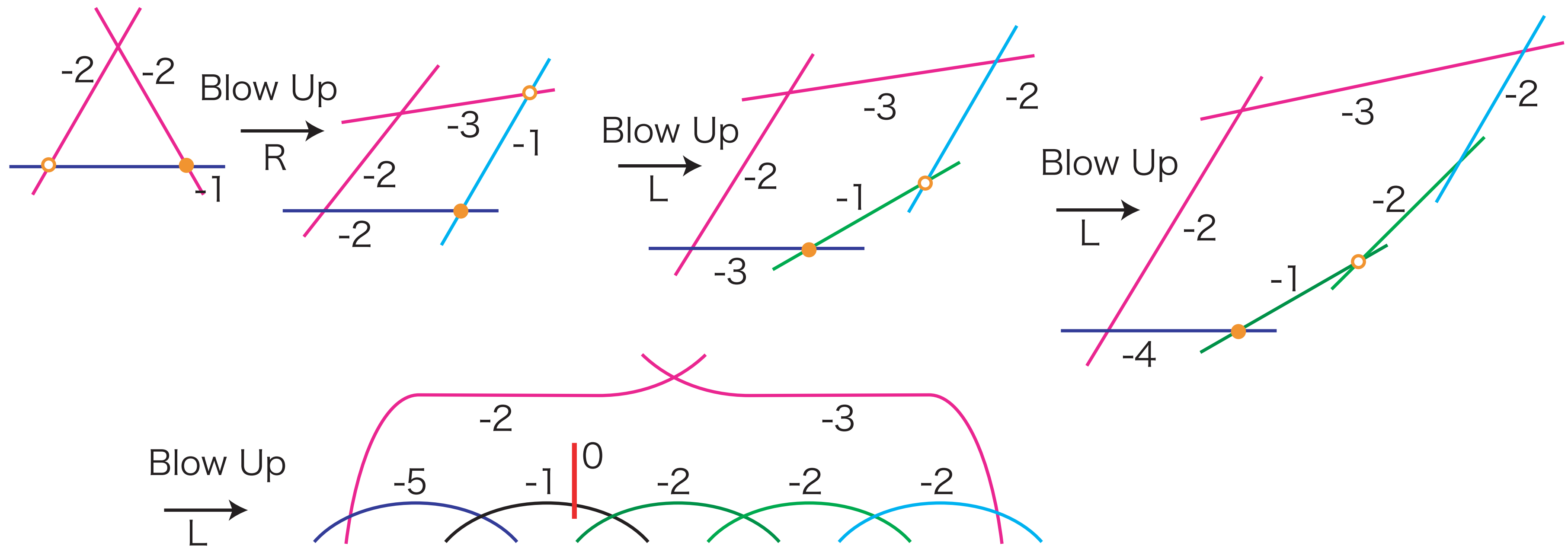


$$[5,2,2,2,2] = \frac{21}{5}$$

に, 次の おまけ を付け加える

さっきの構成 源 に「おまけ(ピンク)」をつけて

$$(4,5)\ \to_R\ (4,1)\ \to_L\ (3,1)\ \to_L\ (2,1)\ \to_L\ (1,1)$$



$$[5,2,3,2,2,2] = \frac{61}{14}$$

$(k^+(4,5);61) = L(61,14)$. $(61 = 4^2 + 4\cdot 5 + 5^2)$

ここから後半です：

Example. $(a,b)=(7,2)$ (corresponding to $(p,q)=(9,2)$)

$(a_i,b_i):\quad (7,2)\to_L(5,2)\to_L(3,2)\to_L(1,2)\to_R(1,1).$,

$(m_i,n_i):\quad (1,1)\to_L(2,1)\to_L(3,1)\to_L(4,1)\to_R(4,5).$

$(s_i,t_i):\quad (1,0)\to_L(1,0)\to_L(1,0)\to_L(1,0)\to_R(1,1).$

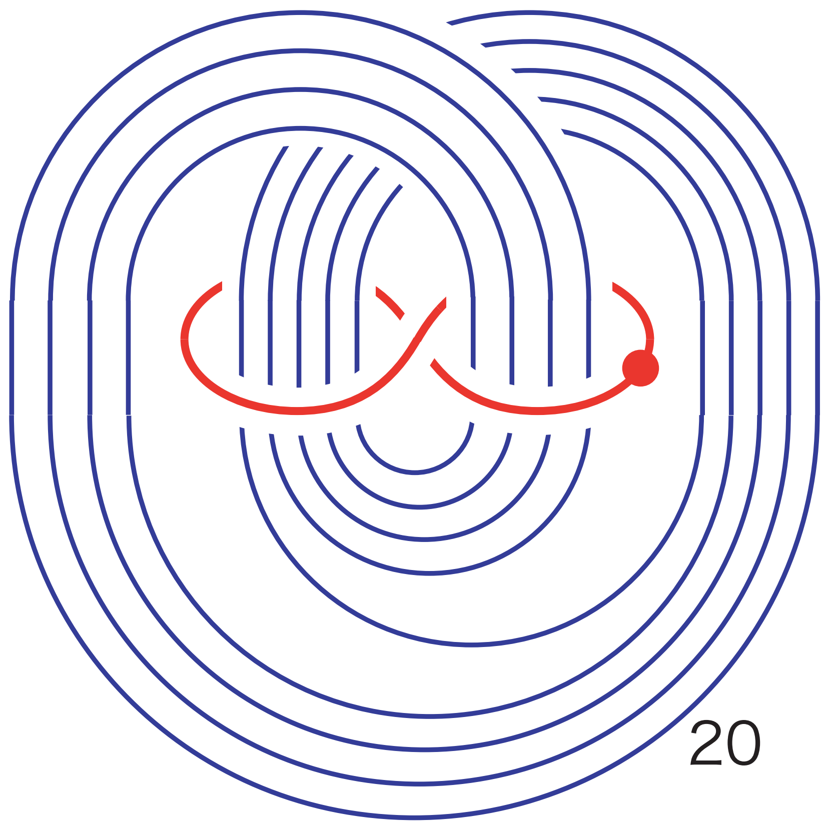
$w(7,2)=LLLR$, $W(7,2)=RLLL$, and $A(7,2)=(4,5)$.

i	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	a_4
0		-1	-1	-1			
1	-1	-2	-1	-2			
2	-1	-3	-1	-2	-2		
3	-1	-4	-1	-2	-2	-2	
4	-1	-5	-1	-2	-2	-2	-2

i	$\overline{a}_{-3}$	$\overline{a}_{-2}$	$\overline{a}_{-1}$	$\overline{a}_0$	$\overline{a}_1$
0				-4	
1				-5	-2
2			-2	-5	-3
3		-2	-2	-5	-4
4	-2	-2	-2	-5	-5

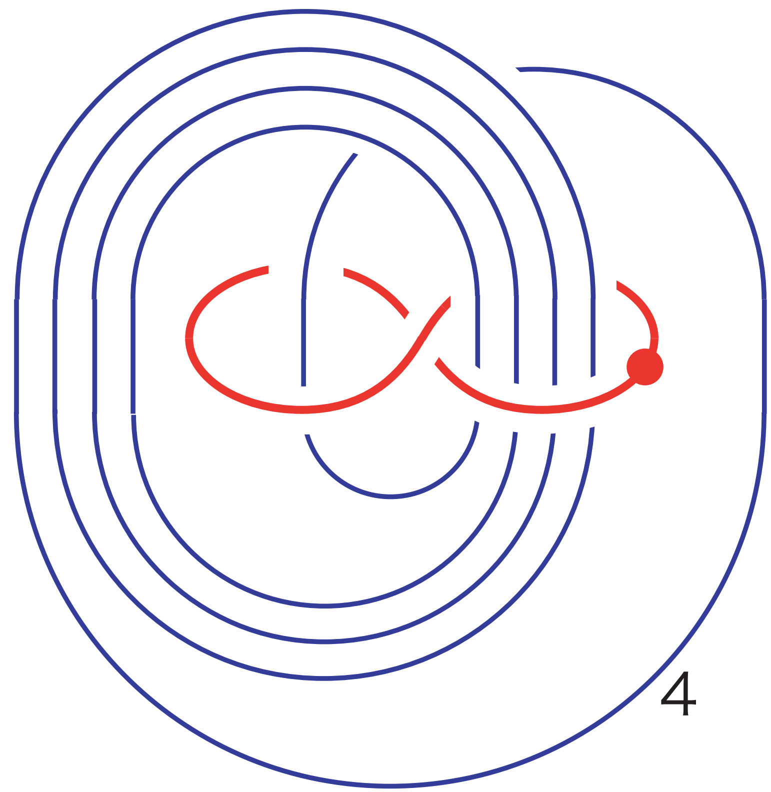
We get the sequence $(-2,-2,-2,-5,-5)$ $(=C_{9,2})$,
and $[5,5,2,2,2]=81/17$.

$(p,q)=(9,2)$ の場合, 前のページのアルゴリズムから,
$(m,n)=A(p-q,q)=A(7,2)=(4,5)$ となるので,



★ non-trivial 成分は Torus knot $T(m,n)$ で framing は mn.

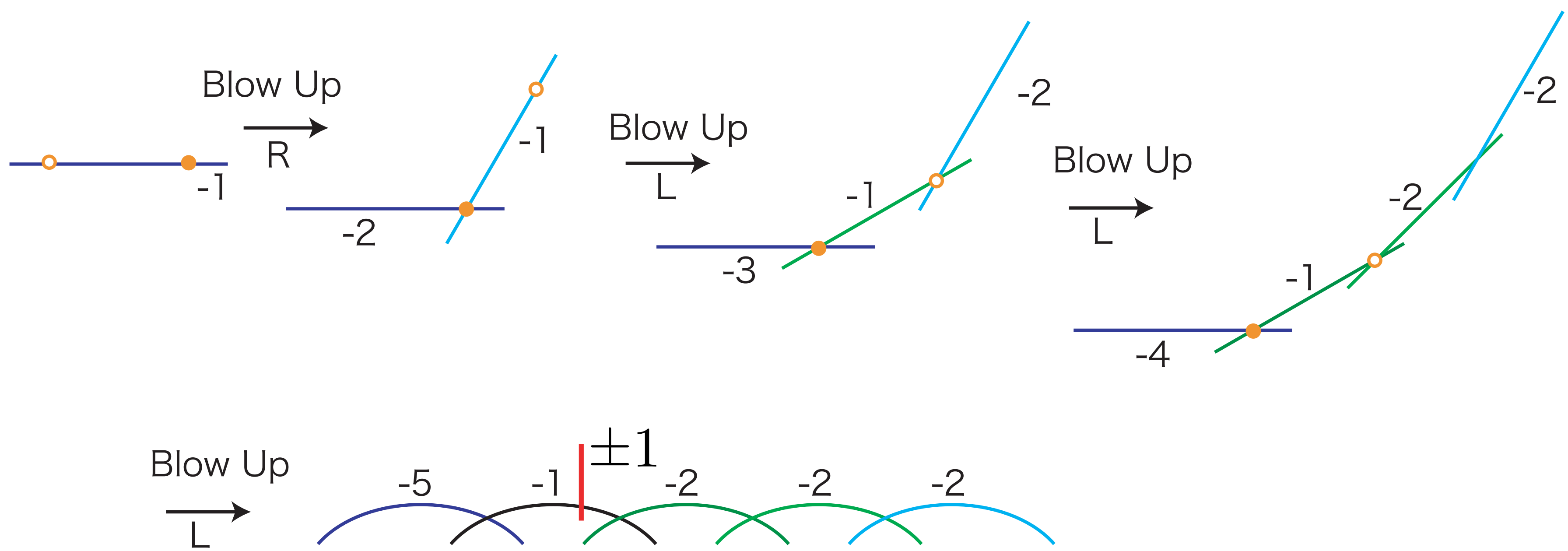
$(p,q)=(5,1)$ の場合,



$X_{5,1}=B_5$ ([Fintushel-Stern]'s rational ball)

今度は, さっきの構成

$$(4,5) \to_R (4,1) \to_L (3,1) \to_L (2,1) \to_L (1,1)$$

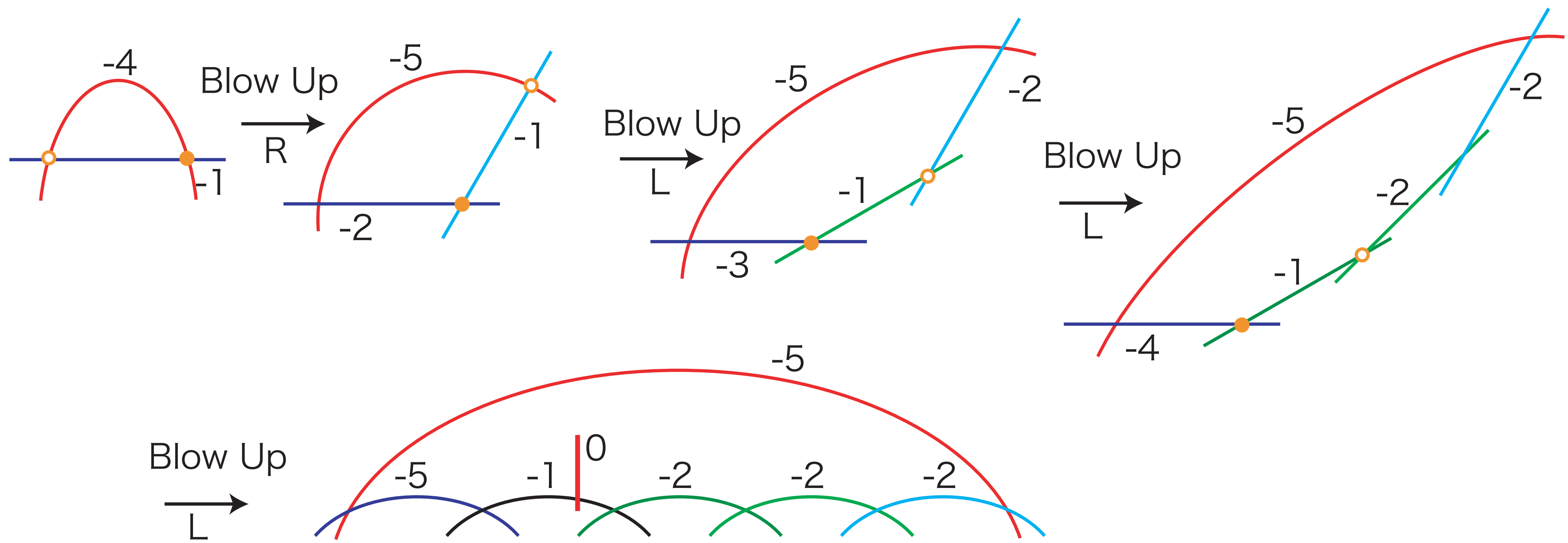


$$[5,2,2,2,2] = \frac{21}{5}$$

に, 別の おまけ を付け加える

2度め の「おまけ(赤)」は

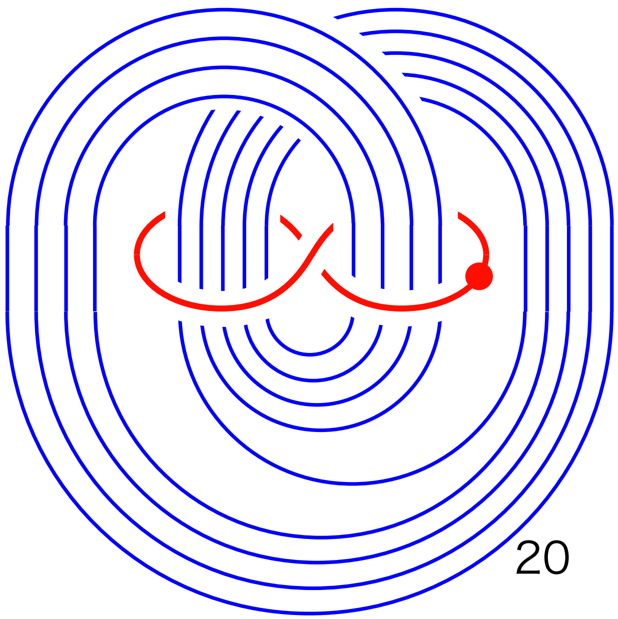
$$(4,5)\ \to_R\ (4,1)\ \to_L\ (3,1)\ \to_L\ (2,1)\ \to_L\ (1,1)$$

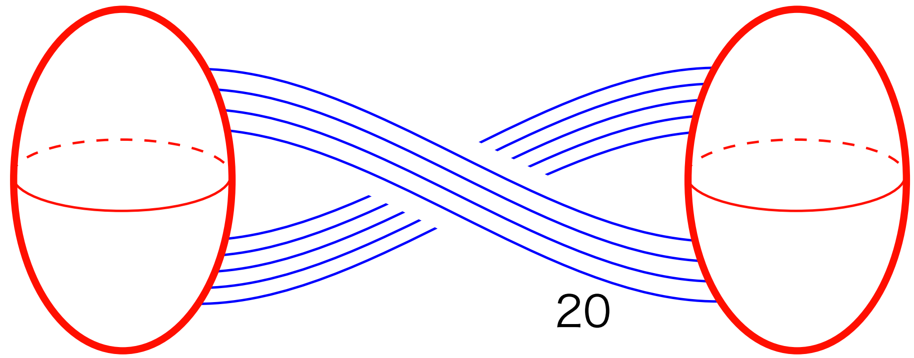


$$[5,5,2,2,2] = \frac{81}{17}$$

$\partial X_{9,2} = L(81,17)$ が示される．$(81 = 9^2\ \ 17 = 9 \times 2 - 1)$

まとめ（$(p,q)=(9,2)$ の場合）





This manifold $X_{9,2}$ satisfies

$$\pi_1(X_{9,2}) \cong \mathbf{Z}/9\mathbf{Z}, \qquad \partial X_{9,2} \cong L(81,17).$$

It is (maybe) a rational ball used in J.Park's "generalized rational blow down".

ありがとうございました