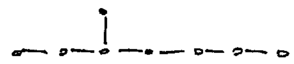
Decomposition of K3 and

Y-M moduli spaces over CH

Tsuyoshi Kato.

M: K3 曲面.



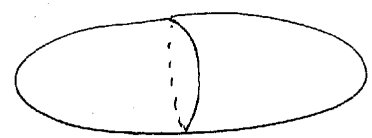
$\varphi : (H_2(M:\mathbb{Z}), \sigma) \cong (\mathbb{Z}^{16} \oplus \mathbb{Z}^6 ; -2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

marking (一意でない)

$3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \sigma(3(S^2 \times S^2))$

問 ∃ 埋め込み ?

$3(S^2 \times S^2) \setminus D^4 \hookrightarrow M$?



答. ① (Donaldson)

No in C^∞

② (Freedman)

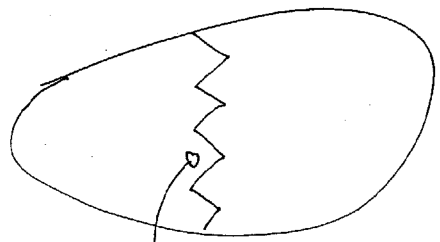
Yes in C^0.

__定理__ (フリードマン)

$\forall \varphi$: marking

$\exists\ \Phi : M \underset{\text{homeo}}{\cong} 2|-E_8| \# 3(S^2 \times S^2)$

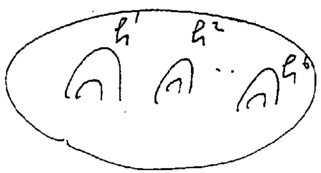
s.t. $\Phi_* = \varphi : H_2 \cong \mathbb{Z}^{16} \oplus \mathbb{Z}^6$



End: Casson handle

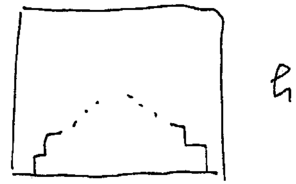
__ハンドル分解__

$$3(S^2 \times S^2) \setminus D^4 = D^4 \cup \underbrace{h^1 \cup \cdots \cup h^6}_{\text{2 handles}}$$



$h^i = (D^2 \times \mathring{D}^2, S^1 \times \mathring{D}^2)$

$CH \subset h$ open subset



- $CH \underset{\text{Top}}{\cong} (D^2 \times \mathring{D}^2, S^1 \times \mathring{D}^2)$

- $\exists\ CH_1, \cdots, CH_6$

$$S = D^4 \cup CH_1 \cup \cdots \cup CH_6 \quad \left(\underset{C^\infty}{\subset} 3(S^2 \times S^2) \setminus D^4 \right)$$

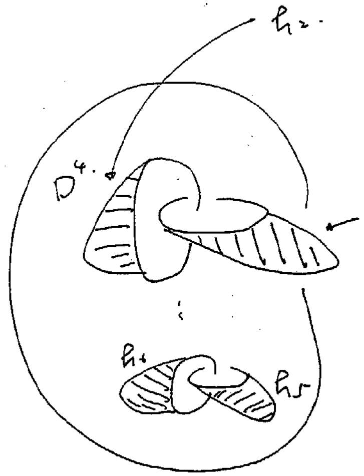
$$\cap\ C^\infty$$

$$M$$

各 CH は. signed ∞ tree T で

あらわされる。

__その構成のアウトライン__



$h_1 = 3(S^2 \times S^2) \setminus D^4$.

By Hurewicz surjection:

$$\pi_2(M) \longrightarrow H_2(M : \mathbb{Z}) \cong \bigoplus_{16} \mathbb{Z} \oplus_6 \mathbb{Z}.$$

$\exists$ C^∞ immersion :

$$H : 3(S^2 \times S^2) \setminus D^4 \looparrowright M$$
$$\underset{h.e.}{\simeq} \; \vee_6 S^2.$$

corresponding to $\bigoplus_6 \mathbb{Z} \cong H_2(3(S^2 \times S^2) : \mathbb{Z})$

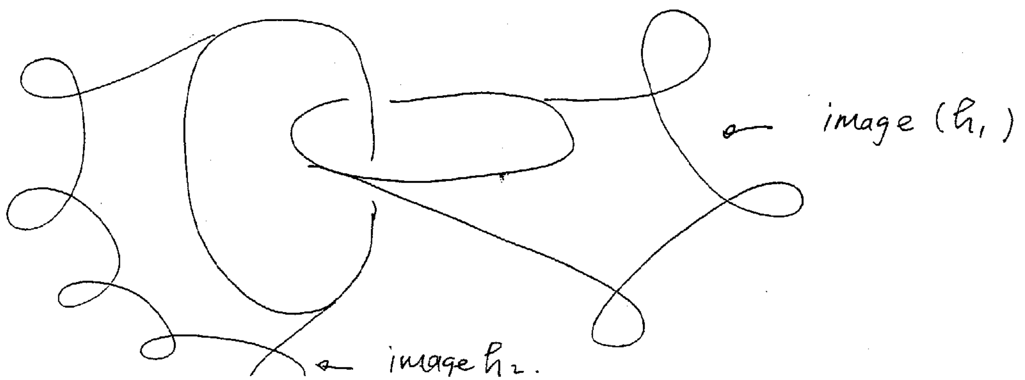
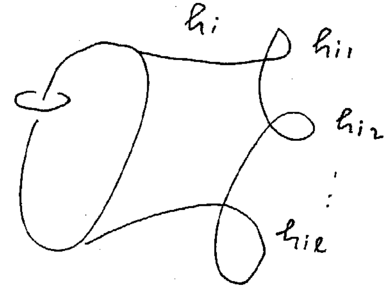


image $h_i \equiv H(h_i) \underset{\text{diff}}{\cong} S^1 \times D^3 \natural \cdots \natural S^1 \times D^3$.

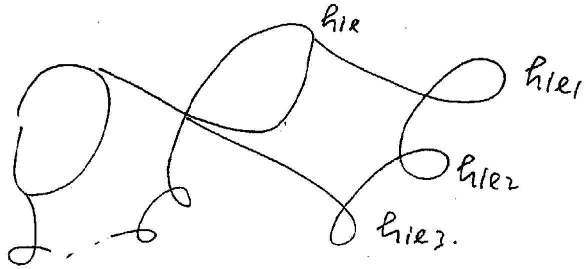
kinky handle.

\# of end connected sums = \# of selfintersections

- In 4 dimension, selfintersection cannot be removed !!
- Instead of capped off h_i and h_j, extra circles ~~occur~~ h_{ij} occur :

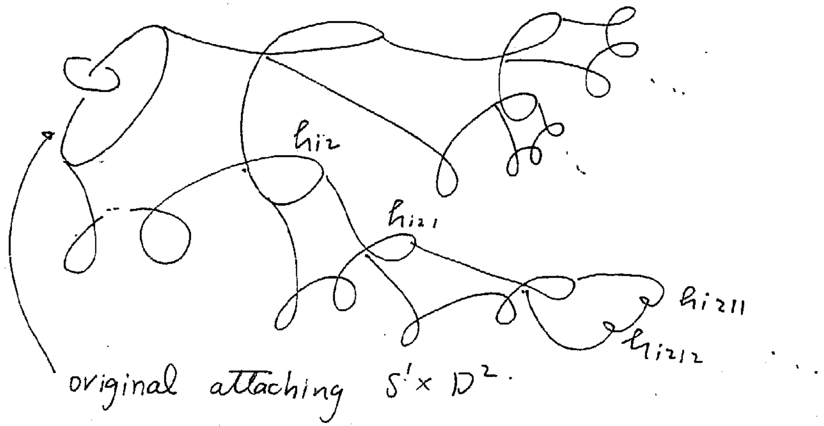


Try to cappe off h_{ik} again by another 2 handles immersed :



◦ h_{ik} has capped off, but extra h_{ikk} occur.

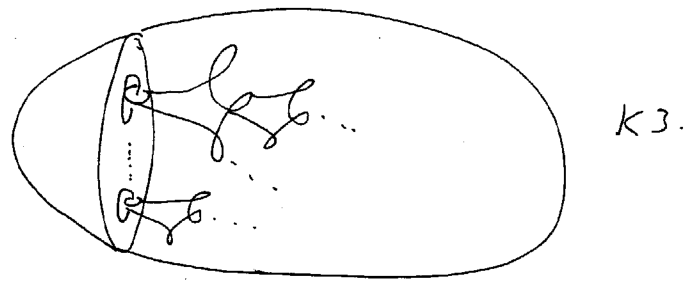
$\Rightarrow$ Continue this process ∞ many times :



$\Rightarrow$ One obtains a tower $(CH, S^1 \times D^2)$ in $K3 \equiv M$ as an open subset.

<u>Th'm</u>. (Freedman)

$$(CH, S^1 \times D^2) \underset{Top}{\cong} (D^2 \times D^2, S^1 \times D^2)$$

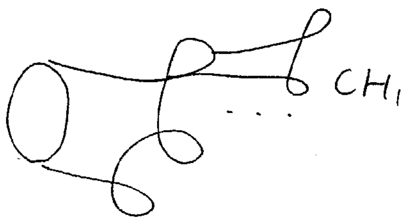


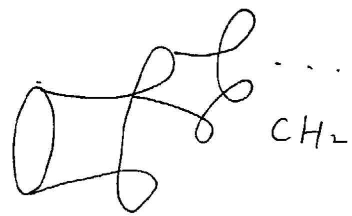
$$S \equiv \underbrace{D^4 \cup CH_1 \cup \cdots \cup CH_6}_{\substack{\text{III} \; Top. \\ 3(S^2 \times S^2) \setminus D^4.}} \subset M.$$

This gives a decomposition of K3 :

$$K3 \underset{Top.}{\cong} 2|-E_8| \# 3(S^2 \times S^2)$$

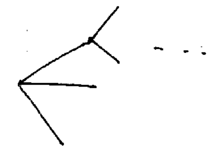
There are various kinds of CH according to the number of selfintersections at each stage :

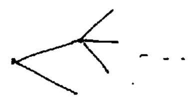




Each CH can be expressed by (signed) trees :

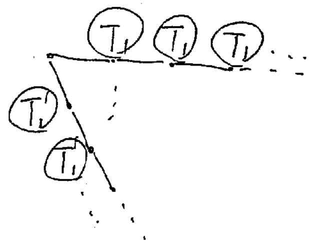
vertex $\longleftrightarrow$ loop.

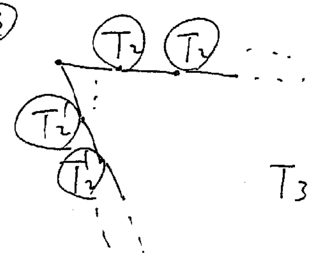
$CH_1 =$  ...

$CH_2 =$  ...

<u>Definition.</u> homogeneous tree of finite type T_n :

①  T_1

②  T_2

③  T_3

⋮

T_n.

T : 有界型 $\underset{def}{\Longleftrightarrow}$ $\exists T_n$: 等質有界型

$T \subset T_n$.

<u>定理 (BG)</u> $\forall$有界型 T, $CH(T)$ はエキゾチック.

<u>主定理</u>

M : K3 曲面.

$\Longrightarrow$ $\exists$ Marking

$$M \underset{Top}{\cong} 3(S^2 \times S^2) \# 2|-E_8|$$

s.t. 対応する $CH_1, \dots, CH_6$ は、すべてが有界型ではあり得ない。

$$\boxed{\text{K3 の分解のエンドの微分構造の複雑さは有界型以上}}$$

特に、周期型ではない。

Remark $T \subset T' \Rightarrow CH(T') \underset{C^\infty}{\subset} CH(T)$

特に、もし $CH(T_1), \cdots, CH(T_6)$ について
$T_1, \cdots, T_6$ すべてが有界型

$\Rightarrow$ $T_1'' \supset T_1, \cdots, T_6'' \supset T_6$ でおきかえて.

$$S = D^4 \cup CH(T_1'') \cup \cdots \cup CH(T_6'') \underset{C^\infty}{\subset} M.$$

として良い

手法 モジュライ理論

- 線型方程式のフレドホルム性 (今日)
- モジュライ空間の形 (明日).

$$S = D^4 \cup CH_1 \cup \cdots \cup CH_6$$

定理 1. CH_i: 有界型 (等質)

$\Rightarrow$ S 上に $\exists$ complete R-metric g of b'dd geom.
$\exists\, w : S \to [0, \infty)$ weight function

s.t.

$$0 \to W_w^{k+1}(S, g) \xrightarrow{d} W_w^{k}(S, \Lambda^1) \xrightarrow{d^+} W_w^{k-1}(S, \Lambda^2_+) \to$$

is of Fredholm. with

$$H^0 = 0$$
$$H^1 = 0$$

$$b_+^2 = 3 \leq \dim H_2 \leq 6 = 2 \times b_+^2$$

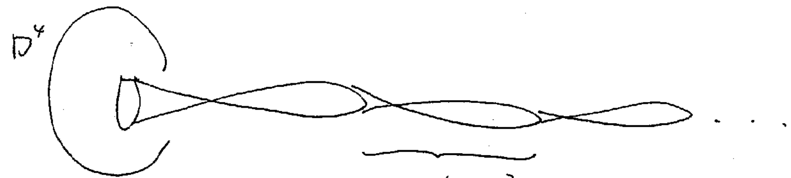
(S, g, w): Admissible $\underset{\text{def.}}{\Leftrightarrow}$ $\begin{cases} ① \text{ AHS complex Fredholm} \\ ② \text{ cohomology groups are computable} \end{cases}$

Admissible pair で OK.

2. Construction of Riemannian metrics and weight functions on Casson handles.

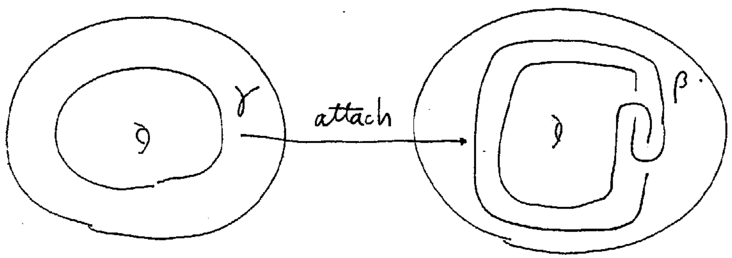
$CH = CH(T)$: finite type $\Leftrightarrow$ T: homog. tree of finite type.

2. A : Periodic Casson handles.



$S^1 \times D^3 \equiv K$: 1-kink.

$\equiv D^4 \underset{S^1\times D^2}{\cup} K_0 \underset{S^1\times D^2}{\cup} K_1 \cup \cdots \qquad (K_i \equiv K)$



$(S^1 \times D^2)_+ \cap \partial K_0$ $\qquad$ $(S^1 \times D^2)_- \cap \partial K_1$.

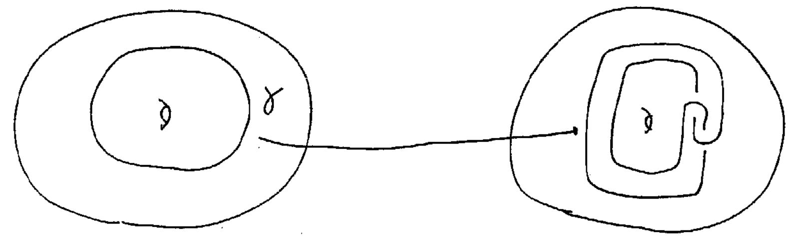
$$\partial K = \partial S^1 \times D^3 = (S^1\times D^2)_+ \cup (S^1 \times D^2)_- = S^1 \times S^2.$$

$\bar{\gamma}$ (tublar nhd in $S^1 \times D^2$) $\cong S^1 \times D^2$

$\bar{\beta}$ (") $\cong$ "

Attach $\bar{\gamma}$ and $\bar{\beta}$ by a diffeomorphism.

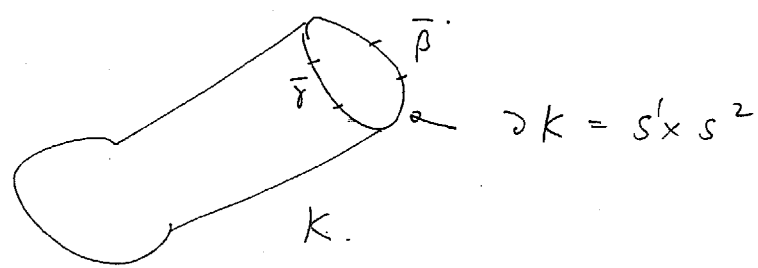
Similarly.



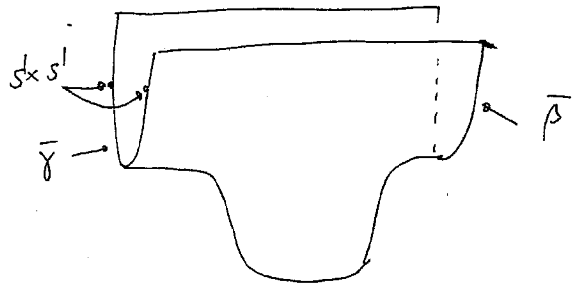
$(S^1 \times D^2)_+ \cap \partial K_1$ $\qquad$ $(S^1 \times D^2)_- \cap \partial K_2$

Construction of a Riemannian metric on $K = S^1 \times D^3$.

<u>Step 1:</u> Choose a cylindrical metric g_0 :

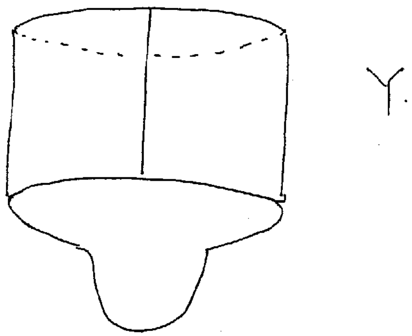


Step 2: Push down $\bar{\gamma}$ and $\bar{\beta}$ ($\cong S^1\times D^2$)

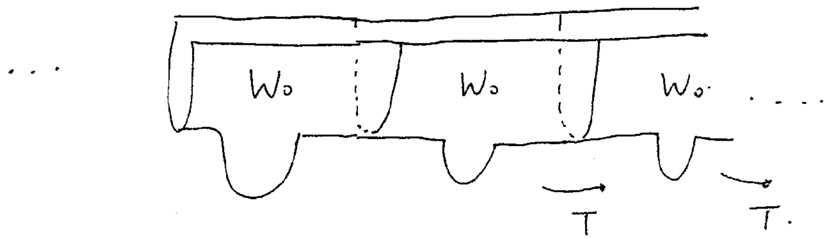


Step 3: Another cylinder manifold:

Identify $\bar{\gamma}$ and $\bar{\beta}$ by an isometry:



Periodic open manifold $\tilde{Y}$: $1\in\mathbb{Z}$ action $\equiv T$.



Y 上の解析の Key point

$\delta>0$. $w: Y \to [0,\infty)$

$w(m,t)=\delta t$, $\operatorname{end} Y \underset{\text{isom}}{\cong} N\times[0,\infty)$

Weighted Sobolev spaces $W^k_w(Y,g)$

$$|u|^2_{W^k_w} = \sum_{\ell\le k}\int_Y e^w|\nabla^\ell u|^2$$

Lift: $\tilde{w}: \tilde{Y}\to[0,\infty)$

$W^k_{\tilde{w}}(\tilde{Y},\tilde{g})$ も同様.

Key Lemma ① $\delta>0$ small $\Rightarrow$

$$0\to W^{k+1}_w(Y,g)\xrightarrow{d} W^k_w(Y,\Lambda^1)\xrightarrow{d^+} W^{k-1}_w(Y,\Lambda^2_+)\to 0$$

acyclic

② acyclic $\Rightarrow$

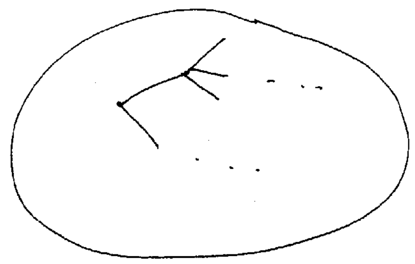
$$0\to W^{k+1}_{\tilde{w}}(\tilde{Y},\tilde{g})\xrightarrow{d} W^k_{\tilde{w}}(\tilde{Y},\Lambda^1)\xrightarrow{d^+} W^{k-1}_{\tilde{w}}(\tilde{Y},\Lambda^2_+)\to 0$$

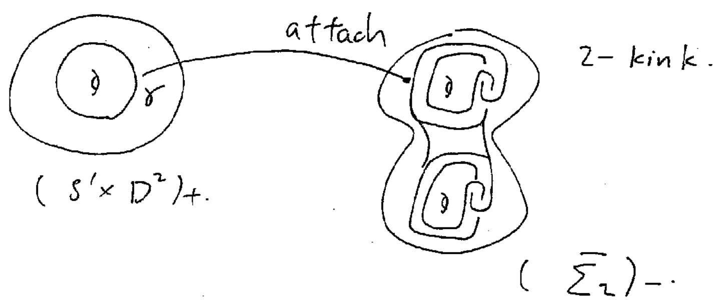
acyclic

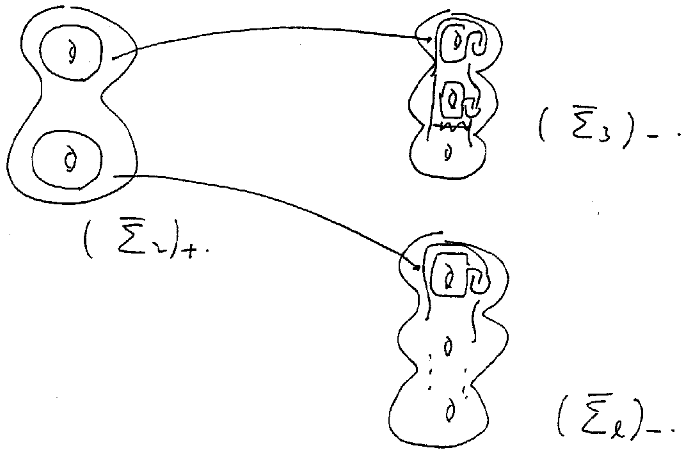
(Taubes F-L transform)

2.3. Generalization

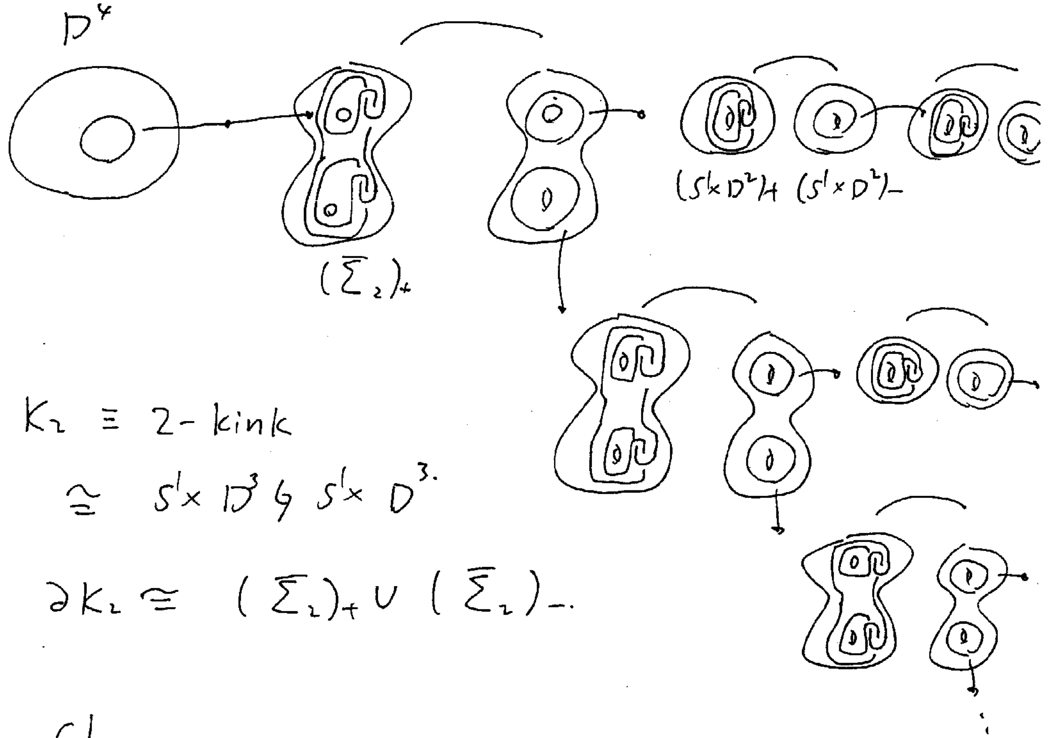
In general CH can be expressed by trees.







Consider simplest T_2:

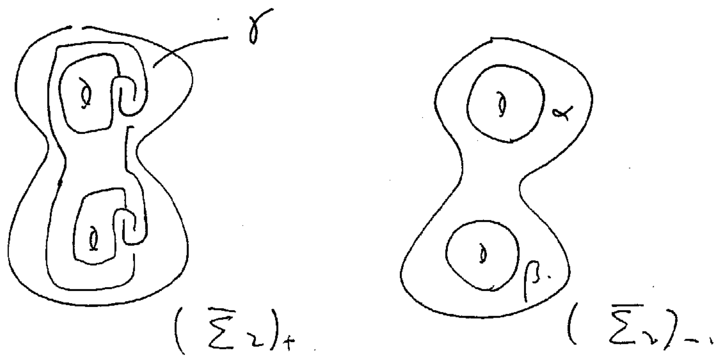


$K_2 \equiv$ 2-kink

$\cong S^1 \times D^3 \natural S^1 \times D^3$.

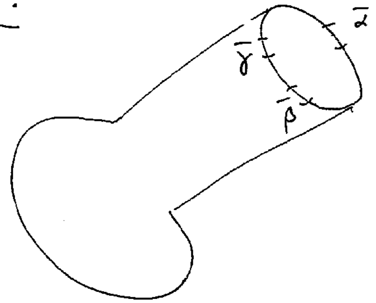
$\partial K_2 \cong (\bar{\Sigma}_2)_+ \cup (\bar{\Sigma}_2)_-$.

Choose



Parallel Construction of R-metric :

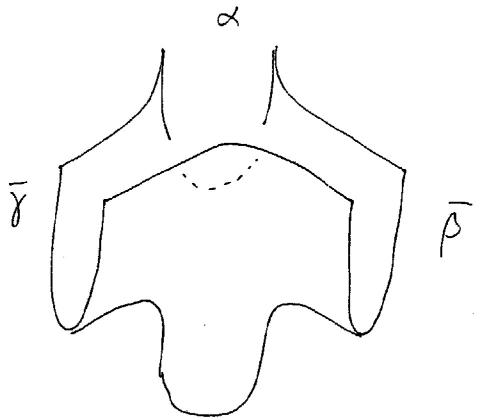
<u>Step 1</u> :



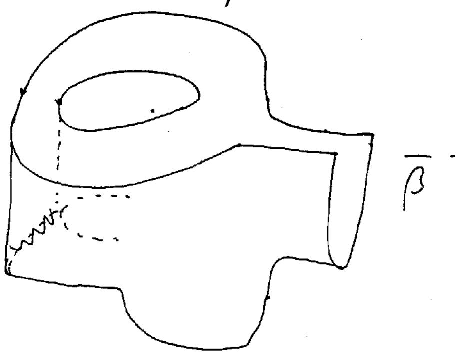
cylinder manifold.

K_2.

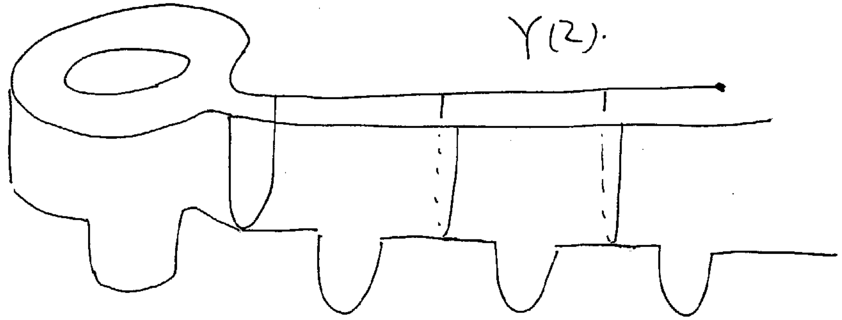
<u>Step 2</u> :



<u>Step 3</u> : Identify $\bar{\gamma}$ and $\bar{\alpha}$

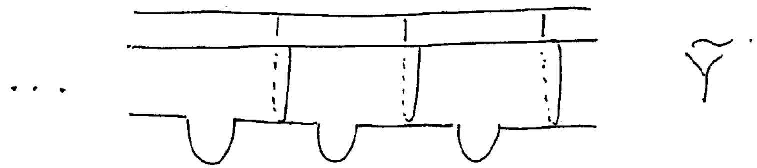


<u>Step 4</u> : Attach a half periodic Casson handle to $\bar{\beta}$



<u>Prop</u> The AHS complex over $Y(2)$ is of null Fredholm.

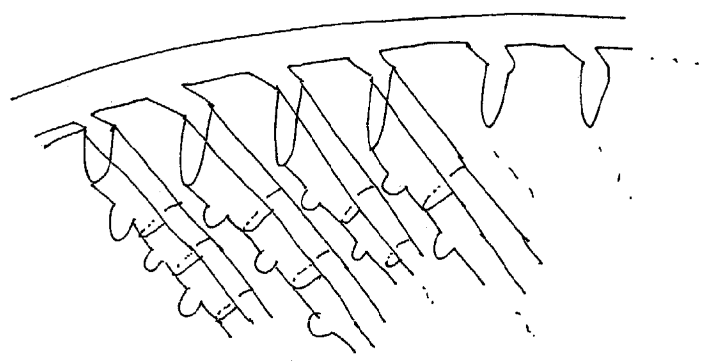
use : ① The AHS complex over :



is of null Fredholm.

② excision.

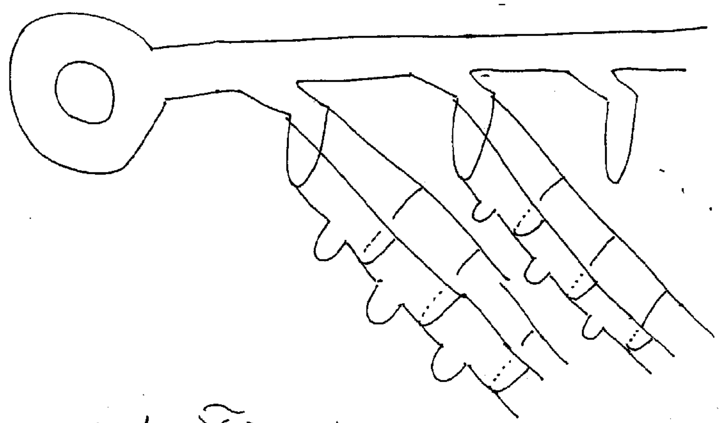
Step 5: Do Fourier-Laplace transform on $\widetilde{Y}(2)$:



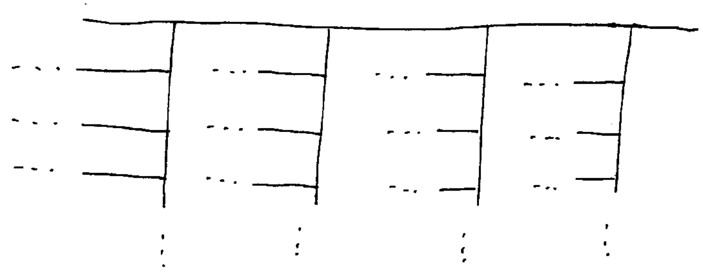
to see the AHS complex over $\widetilde{Y}(2)$ is of null Fredholm.

Step 6: One more higher case:

Similar procedure for $Y(3)$:



and $\widetilde{Y}(3)$ becomes:



$\widetilde{Y}(3)$