§3. Moduli Spaces

3. A. Yang - Mills moduli spaces for closed M.

M: C^∞ closed oriented 4 manifold.

$E \to M$ $SO(3)$ bundle $\Leftrightarrow$ $(w_2(E), p_1(E))$.

A_0: C^∞ reference connection.

$\mathcal{A} \equiv \{ A_0 + a : a \in \Gamma(M ; \mathrm{Ad}E \otimes \Lambda^1) \}$

$\mathcal{G} \equiv \Gamma(\mathrm{Aut}\, E)$.

(In precise we use Sobolev spaces)

action $\mathcal{G} \curvearrowright \mathcal{A}$ by $g^* A = g^{-1} \circ A \circ g$.

Fix a Riemannian metric g on M

$\Rightarrow$ $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$

$\ni$ $*a = a$ selfdual

__Definition__. A: ASD connection $\Leftrightarrow$

F_A is ASD (ie. $F_A^+ \equiv F_{A}^{+} = 0$,

$\hat{\mathcal{M}}(E, (M, g)) = \{ A \in \mathcal{A} : \mathrm{ASD} \}$

$\mathcal{G} \curvearrowright \hat{\mathcal{M}}$,

$\mathcal{M}(E, (M, g)) \equiv \hat{\mathcal{M}} / \mathcal{G}$: ASD moduli sp

__Rem__ ① $\mathcal{M}(E, (M, g))$ metric dependent

② If $\#\mathcal{M}(E, (M, g)) \neq 0$

$\Rightarrow$ $\forall$ generic g', $\mathcal{M}(E, (M, g')) \neq \phi$.

__線形化方程式__

$0 \to W^{k+1}(M, \mathrm{Ad}E) \xrightarrow{d_A} W^k(M, \Lambda^1 \otimes \mathrm{Ad}E) \xrightarrow{d_A^+} W^{k-1}(M, \Lambda^2_+ \otimes \mathrm{Ad}E) \to 0$

$T_{[A]}\mathcal{M} \cong \dfrac{\mathrm{Ker}\, d_A^+}{\mathrm{im}\, d_A} = H_A^1$ (generic)

__Thm__ (F-U) g: generic metric

$\Rightarrow$ $\mathcal{M}$ is a C^∞ mfd of dim.

$-2 p_1(E) - 3(1 - b_1(M) + b_2^+(M))$

Setting.

① M: K3 曲面.

$\begin{cases} E \to M & SO(3) \text{束}. \\ {}^{\forall}\text{generic } g, & \mathcal{M}(E,(M,g)) \neq \phi. \end{cases}$

② Choose a marking $\varphi : (H_2(M,\mathbb{Z}),\sigma) \cong (\mathbb{Z}^{16}, -E_8) \oplus (\mathbb{Z}^6, 3H)$
so that.
$S = D^4 \cup CH_1 \cup \cdots \cup CH_6 \subset M$

有界(等質)型.

$\Rightarrow$ ${}^{\exists}$ admissible pair (g, w). fix.

Convergence of ASD connections

Take

- $K_0 \subset\subset K_1 \subset\subset \cdots \subset S \subset M$.
- h_i: R-metrics on M s.t. $\underline{h_i|K_i = g|K_i}$

$\Rightarrow$ ${}^{\forall}[A_i] \in \mathcal{M}(E,(M,h_i))$

$A_i|K_i$: ASD wrt g.

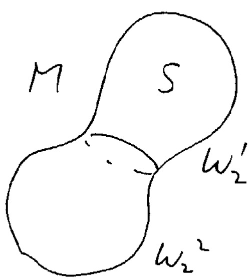
Proposition (U)

${}^{\exists}$ subsequence $A_{k_i} \to A_\infty$: ASD over ${}^{\exists}E' \to (S,g)$.

$k = -p_1(E) = \frac{1}{2\pi}\int_M |F_{A_i}|^2$

$\vee\!\mid$

$k' = -p_1(E') = \frac{1}{2\pi}\int_S |F_{A_\infty}|^2.$



Key observation $w_2 = w_2' \oplus w_2^2$ wrt φ.

- If $w_2^2 \neq 0 \Rightarrow k \geq k' + 2$.
- $k' > 0$ if $w_2' \neq 0$.

ASD moduli spaces over S

$\mathcal{A}(A_\infty) = A_\infty + W^{k+1}_w(S, \Lambda^1 \otimes \mathrm{ad}\, E)$.

$\mathcal{G} = \mathrm{Aut}\, E' \cap \{u \quad : u - id \in W^{k+2}_w\}$.

$\hat{m}(A_\infty) = \{ A \in \mathcal{A}(A_\infty) : \text{ASD} \}$.

$$\boxed{m(A_\infty) = \frac{\hat{m}(A_\infty)}{\mathcal{G}}.}$$

③ $m(A_\infty)$: regular $\underset{\text{def}}{\Longleftrightarrow}$ $\forall [A] \in m(A_\infty)$

$d_A^+ : W_\omega^{k+1} \to W_\omega^{k}$ 全射

$\Longrightarrow$ $m(A_\infty)$ is a C^∞ mfd of

dim = index AHS

$\leqq \dim m(E, M) - 4$ (if $w_2 \neq 0$)

④ $m(A_\infty) \neq \emptyset$.

$\Longrightarrow$ $\dim m(E, M) = 0$ のとき

①〜④ すべては満たされない。

<u>F-U generic perturbation of R-metric</u>

<u>works for this case :</u> 摂動における困難.

C : {the set of small perturbations of g on S.}

$\Longrightarrow$ $\exists\ \mathcal{B} = \mathcal{B}(A_\infty) \subset C$ s.t. $\forall g' \in \mathcal{B}$.

- $M(A_\infty, g')$ is regular if non empty
- (g', w): admissible.

<u>Try the same procedure using (g', w):</u>

- $K_0 \subset\subset K_1 \subset\subset \cdots \subset S \subset M$.
- h_i' : R-metrics on M $h_i'|K_i = g'|K_i$

$[A_i'] \in \mathcal{M}(E, (M, g_i'))$

$\Downarrow$ subsequence

A_∞' : ASD over (S, g')

In general,

$[A_\infty'] \notin \mathcal{M}(A_\infty, g')$

since $A_\infty' - A_\infty \in L^2(S, \Lambda^1 \otimes \mathrm{ad}E)$

$\notin L^2_w(\quad \text{''} \quad)$

In weighted Sobolev spaces,

$$\begin{array}{ll} A_\infty \cdot & \cdot A_0' \\ A_1 \cdot & \cdot A_1' \\ A_2 \cdot & \cdot A_2' \\ A_i \cdot & \cdot A_i' \\ \vdots & \ddots \\ A_\infty \cdot & \cdot A_\infty' \end{array}$$

一致. g' は A_∞' について generic ではない?

$\mathcal{M}(A_\infty', g')$: non regular

M: K3 曲面.

$\varphi : (H_2(M : \mathbb{Z}), \sigma) \cong (\mathbb{Z}^{16} \oplus \mathbb{Z}^6, -2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

marking

$E \to M$ SO(3)束 $\Leftrightarrow$ $p_1(E)$, $w_2(E)$

φ について. $w_2 = w_2^1 + w_2^2$.

Def. (E, φ): generic $\Leftrightarrow$ $w_2^1, w_2^2 \neq 0$.

$\mathcal{M}(E)$: ASD moduli space over $E \to M$.

$\dim \mathcal{M}(E) = 0 \Rightarrow$ $Q(E) = \# \mathcal{M}(E) \in \mathbb{Z}$.

Donaldson invariant:

metric dependent

smooth invariant (metric independent)

Lemma (Kronheimer) $\boxed{\text{Existence}}$

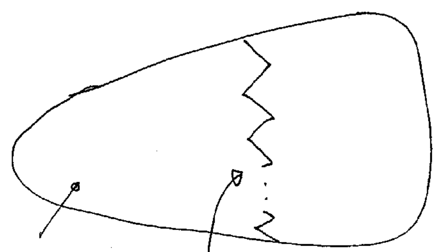
$\exists E \to M$, $-p_1(E) = 6$.

$(E, \exists\varphi)$: generic marking

s.t. $Q(E) = \pm 1$.

Casson handles

marking $\Longrightarrow$



$\mathrm{End} = CH_1 \cup \cdots \cup CH_6$.

主定理

$\boxed{\begin{array}{l} M:\ K3\text{ 曲面}. \\ E \to M \quad SO(3)\text{ 束} \\ \circ\ Q(E) \neq 0 \\ \Longrightarrow \forall\ \text{generic marking について. 対応する} \\ \quad CH_1, \cdots, CH_6\text{ は. すべてが有界型} \\ \quad \text{ではあり得ない。} \end{array}}$

Rem CH: 有界型 $\Longrightarrow$ $\exists CH' \subset_{C^\infty} CH$ 等質有界型.

以下. $Q(E) \neq 0$ の下. $\exists$ generic marking について. $CH_1, \cdots, CH_6$ すべてが等質有界型として矛盾を導く.

定理 1 $S = D^4 \cup CH_1 \cup \cdots \cup CH_6$

等質有界型

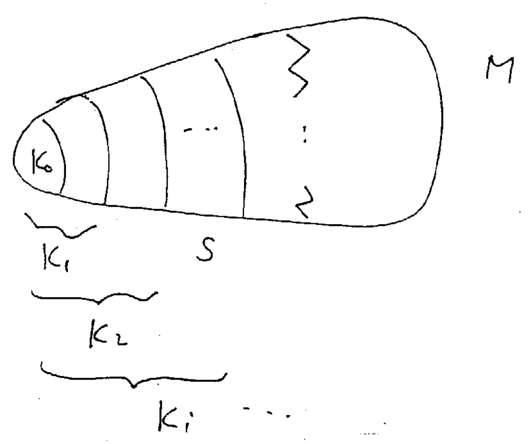
$\Longrightarrow\ ^\exists (g, w)$ b'dd geom. s.t.

$$0 \to W^{k+2}_w(S, g) \xrightarrow{d} W^{k+1}_w(S, g : \Lambda^1) \xrightarrow{d^+} W^k_w(S, g : \Lambda^2_+) \to 0$$

Fredholm

$H^0 = 0$, $H^1 = 0$, $\dim H^2 \geqq 3$.

Metric deformation

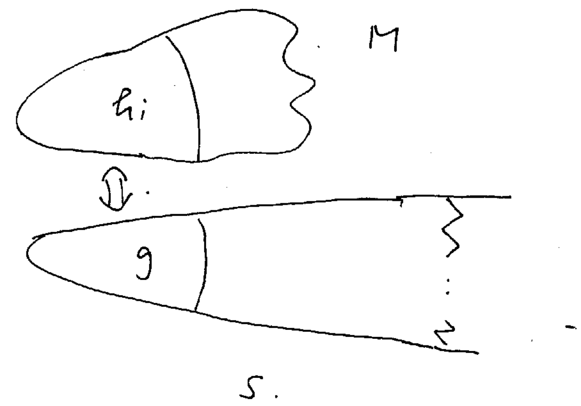


$$K_0 \subset\subset K_1 \subset\subset \cdots \subset\subset K_i \subset\subset \cdots \subset S \subset M$$

Exaustion by compact subsets.

$\{(M, h_i)\}_{i=0}^\infty$ generic family of R-metrics

s.t. $h_i | K_i \to g | K_i$ in C^∞.



$Q(E) \neq 0 \Longrightarrow\ ^\exists [A_i] \in \mathcal{M}(E : (M, h_i))$

$A_i \longrightarrow A : \begin{cases} (S, g) \text{ について ASD.} \\ F_A \in L^2(S, g : \Lambda^2 \otimes E). \end{cases}$

Lemma. $-p_1(E) = 6 \Longrightarrow\ ^\forall$ generic marking について A_i はバブルしない。

∵ 特性数のギロン

YM moduli space over (S, g, w)

A : L^2 ASD connection fix.

$\mathcal{A}(A) = A + W_w^{k+1}(S, g : \Lambda^1 \otimes \mathrm{ad}E)$

$\mathcal{G} = \{ h \in \mathrm{Aut}(E) : h - id \in W_w^{k+2} \}$.

$$\boxed{\mathcal{M}(A) = \{ A' \in \mathcal{A}(A) : \text{ASD over } (S, g) \} / \mathcal{G}.}$$

形式的有限次元

A : regular $\underset{\text{def}}{\Leftrightarrow}$ $d_A^+ : W_w^{k+1} \twoheadrightarrow W_w^k$ 全射

$\Longrightarrow$ ムジュンがおこる

$\because \begin{cases} \circ \ \dim \mathcal{M}(A) < 0 \\ \circ \ [A] \in \mathcal{M}(A) \end{cases}$

以下. 全射を仮定しないで. うまい摂動をとることでムジュンを導く.

F-U perturbation of metrics

$C = \{$ the set of small perturbation of $g \}$

$\Longrightarrow \ ^{\exists} \mathcal{B}(A) \subset C$: dense s.t. $^{\forall} g'$.

$$\boxed{\mathcal{M}(A, g') : \text{regular mfd or } \phi.}$$

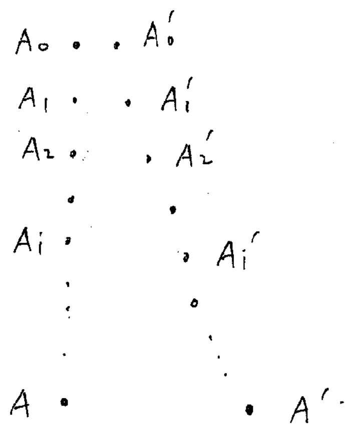
$\begin{cases} h_i : & \text{generic } R\text{-metrics on } M \\ h_i' : & \quad '' \end{cases}$

$\begin{cases} h_i | K_i \to g \\ h_i' | K_i \to g' \end{cases}$

$\boxed{\text{parallel procedure}}$

$\begin{cases} [A_i] \in \mathcal{M}(E, (M, h_i)) \\ [A_i'] \in \mathcal{M}(E, (M, h_i')) \end{cases}$

$\Longrightarrow \begin{cases} A_i \longrightarrow A & \text{over } (S, g) \\ A_i' \longrightarrow A' & \text{over } (S, g'). \end{cases}$



一般に $A - A' \notin W^{k+1}_{w}(S, g; \Lambda^1 \otimes \mathrm{ad}E)$

$\Rightarrow$ $A' \notin \mathcal{A}(A)$ より

$\boxed{[A'] \notin M(A, g')}$

うまい摂動をとることで $\in$ とできることを示す。

$\boxed{\text{一様局所座標系の構成}}$

<u>Local な perturbation</u> 像が $\underbrace{S(0, \mathcal{B})}$

$\mathcal{B}$: Banach manifold (あとで ∞ dim が必要)

A : $X = M$ or S 上の connection.

$S : \mathcal{B}(X) \times \mathcal{B} \to W^k_{loc}((X, g); \Lambda^2_+ \otimes E)$

C^∞ map

s.t. $S(A, b)$ は $A|K_0$ のみに depend

であって、Moduli space 上の 本性 machinery が行えるものを1つ fix.

ex: Floer の holonomy perturbation.

perturbed ASD moduli space

$b \in \mathcal{B}$.

$$M_b(A)_w = \{A' \in A + W^{k+1}_w(S, g) : \overbrace{F^+(A') + s(A', b)}^{F^+_b(A')} = 0\} / \mathcal{G}$$

$M_b(M, g_i)$ も同様.

b : generic で これらは 有限次元多様体 か ϕ.

以下. $w_i : M \to [0, \infty)$ ファミリーの weight f.c.n. s.t. $w_i | K_i = w | K_i$ を fix.

<u>Rem</u>

M: compact $\Rightarrow$ Sobolev space $W^*(M, h_i) \to W^*_{w_i}(M, h_i)$ に変えても $\mathcal{M}(E, (M, h_i))$ は同じ

以下. 関数空間は. $\underline{W^k_{w_i}(E, M)}$ を用いる。

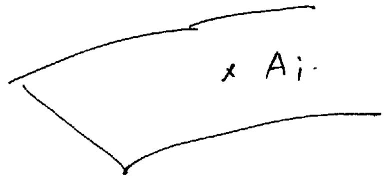
$d^+_{A_i} : W^{k+1}_{w_i}(M, h_i : \Lambda^1 \otimes \mathrm{ad}E) \twoheadrightarrow W^k_{w_i}$ $\underline{\underline{全射}}$. であった。

$(d^+_{A_i})^*_{w_i} : W^{k+1}_{w_i} \to W^k_{w_i}$ weighted adjoint operator

<u>∞ dim implicit function theorem</u>

$$\mathcal{A}(E) = A_i + W^{k+1}_{w_i}((M, h_i); \Lambda^1 \otimes \mathrm{ad}E)$$

Affine space



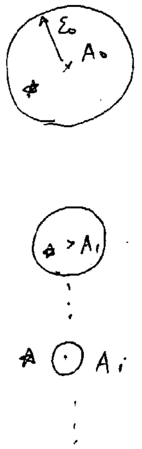
A_i: regular $\Rightarrow \| (d^+_{A_i})^*_{w_i} \| \geq {}^\exists C_i > 0$

$$\left\{ \begin{array}{l} \exists\, \varepsilon_i > 0, \quad N^i_1 \times N^i_2 \subset B_{\varepsilon_i}(A_i). \\ \exists\, G_i : N^i_1 \to N^i_2 \quad C^\infty \end{array} \right.$$

s.t. $\forall A' = (a, b) \in N_1 \times N_2$.

$$\boxed{F^+_b(A') = 0 \iff b = G(a)}$$

<u>Point</u>. $\left\{ \begin{array}{l} \circ\ \varepsilon_i \text{ は. } C_i \text{ に depend} \\ \circ\ \| dG_i \| \text{ も } \quad \text{〃} \\ \quad (\text{〃} \ -(d^+_{A_i})^{-1} \circ d^+_{A_i} \ ) \end{array} \right.$



☆ : solutions.

$C_i \to 0 \Rightarrow \varepsilon_i \to 0$

$\boxed{\text{解が local chart の中で見つけられなくなる!}}$

Test Case

$\|(d^+_{A_i})^*_{w_i}\| \geq C > 0$ 一様評価

が成り立つと仮定すると

$\Rightarrow \ \exists [B_i] \in \mathcal{M}_b(E, (M, h_i))$

$\|A_i - B_i\|_{W^{k+1}_{w_i}(M, h_i)} \leq \exists C'$

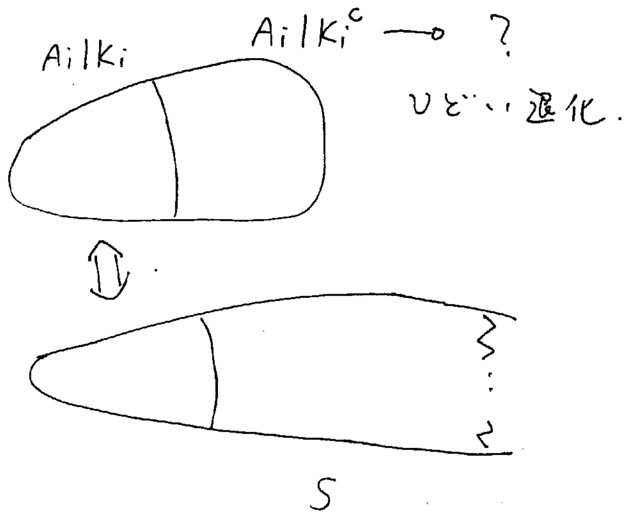
$\downarrow \qquad \downarrow$

$\|A - \exists B\|_{W^{k+1}_{w}(S, g)} \leq C.$

$\Rightarrow \ [B] \in \mathcal{M}_b(A)$ となり、証明はおわる。

(cf. Last step of the proof)

一般にこの状況は期待できない。



しかし $A_i|K_i^c$ を"消す"方法があれば良い。

以下 $\left(\begin{array}{ll} (S, g, w), & [A] \\ \{(M, h_i, w_i)\}_{i=0}^{\infty}, & [A_i] \end{array}\right)$ を fix.