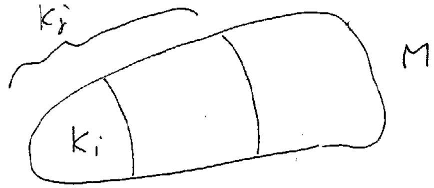
$\boxed{\text{商ソボレフ空間}}$



$j \geq i$.

$$W^*_{w_i}(K_i^c, h_j)_0 = \text{closure of } C_c^\infty(K_i^c)$$

$$\cap \quad \text{closed}$$

$$W^*_{w_i}(M, h_j)$$

<u>直交分解</u>

$$W^*_{w_j}(M, h_j) = W^*_{w_i}(K_i^c, h_j)_0 \oplus (\quad)^\perp.$$

<u>商ソボレフ空間</u>

$$\overline{W}^*_{w_j}(K_i, h_j) \equiv W^*_{w_j}(M, h_j) / W^*_{w_j}(K_i^c, h_j)_0$$

(ノルムは直交分解を用いる)
h_j に depend

以下. $\overline{W}^*_{w_j}(K_i)$ を用いてこれまでの手続きを再構成する。

$$S : \mathcal{A}(E) \times \mathcal{B} \longrightarrow W^k_{w_j}(M, h_j) : \Lambda^2_+ \otimes \mathrm{ad}E,$$

<u>Local Perturbation</u>

$$F^+_S : \mathcal{A}(E) \times \mathcal{B} \to W^k_{w_j}$$

$$F^+_S(A_j + \alpha, b) \equiv F^+(A_j + \alpha) + S(A_j + \alpha, b)$$

$\Longrightarrow$ 誘導 $\quad \overline{F}^+_S : A_j + \overline{W}^{k+1}_{w_j}(K_i, h_j) \times \mathcal{B} \to \overline{W}^k_{w_j}(K_i, h_j)$

特に. $\begin{cases} \alpha \in W^{k+1}_{w_j}(M, h_j) \\ \overline{\alpha} \in \overline{W}^{k+1}_{w_j}(K_i, h_j) \end{cases}$

$$F^+_S(\overbrace{A_j + \alpha}^{A}, b) = 0 \Longrightarrow \overline{F}^+_S(\overbrace{A_j + \overline{\alpha}}^{\overline{A}}, b) = 0$$

<u>系</u>. $\quad d^+_{(A,b)} : W^{k+1}_{w_j}(M, h_j) \twoheadrightarrow W^k_{w_j}(M, h_j)$ 全射

$\Longrightarrow \quad d^+_{(\overline{A},b)} : \overline{W}^{k+1}_{w_j}(K_i, h_j) \twoheadrightarrow \overline{W}^k_{w_j}(K_i, h_j)$ 全射.

$d^+_{(A,b)} : W^{k+1}_{w_j}(M, h_j) \twoheadrightarrow W^{k}_{w_j}(M, h_j)$ (係数略)

全射とする.

$\Longrightarrow W = (\mathrm{Ker}\, d^+_{(A,b)})^\perp$ Ⅰ. $\exists C_j > 0$.

$\| d^+_{(A,b)}(u) \|_{W^k_{w_j}} \geq C_j \| u \|_{W^{k+1}_{w_j}}$.

<u>Key lemma</u>

$$\begin{cases} d^+_{(\bar{A},b)} : \bar{W}^{k+1}_{w_j}(K_i, h_j) \twoheadrightarrow \bar{W}^{k}_{w_j} \\ \bar{W}^j = (\mathrm{Ker}\, d^+_{(\bar{A},b)})^\perp \end{cases}$$

について, Ⅰと<u>同じ</u>定数 C_j について.

$\| d^+_{(\bar{A},b)}(\bar{u}) \|_{\bar{W}^k_{w_j}} \geq C_j \| \bar{u} \|_{\bar{W}^{k+1}_{w_j}}$

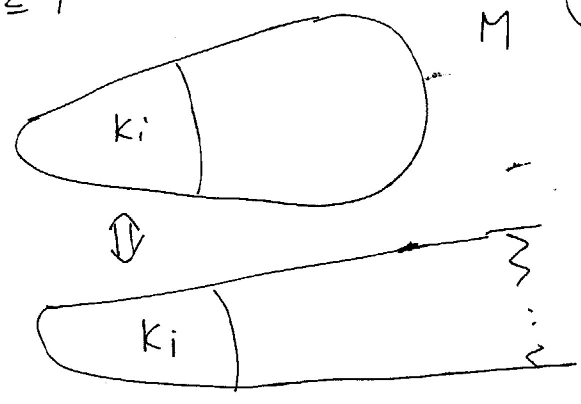
が成り立つ.

<u>Rem</u>. Ⅰは. 一般の作用素で成立.

関数空間の収束

$h_j \to g$
$j \to \infty$

$j \geq i$



$i, j \to \infty$ で $\bar{W}^*_{w_j}(K_i, h_j) \longrightarrow W^*_w(S, g$ "収束"

$j \geq i$.

$\bar{V}_i^j \subset \bar{W}^*_{w_j}(K_i, h_j)$: 線形部分空間のファミリ.

$V \subset W^*_w(S, g)$ closed linear

<u>Def.</u> $\{\bar{V}_i^j\}_{i,j} \Rightarrow V$: <u>∞遠でVに含まれる</u>

$\overset{\Longleftarrow}{\text{def}}$ $\forall \varepsilon > 0$.
$i \geq \forall i_0$ について :

$\bar{W}^*_{w_j}(K_i, h_j) \longrightarrow \bar{W}^*_{w_j}(K_{i_0}, h_j)$
(distance decreasing) projection.

$$\boxed{\begin{array}{l} \forall \{\bar{u}_i^j\}_{i,j} : \text{有界列}, \quad \bar{u}_i^j \in \bar{W}^*_{w_j}(K_i, h_j) \\ \exists v_i^j \in W^*_w(S, g) \cap V \\ \| \bar{u}_i^j - \bar{v}_i^j \|_{W^*_{w_j}(K_{i_0}, h_j)} < \varepsilon. \\ \qquad \text{for all large } j \geq i \gg i_0 \end{array}}$$

$\{\bar{V}_i^j\}_{i,j} \Rightarrow V^\perp$: <u>∞遠でVに直交</u>

$\overset{\Longleftarrow}{\text{def}}$ $\forall \{\bar{u}_i^j\}_{i,j} \in \{\bar{V}_i^j\}_{i,j}$ 有界列
$\forall \{v_i^j\}_{i,j} \subset V$ 〃
$\forall \varepsilon > 0$.

$\Rightarrow \exists i' \gg \forall i_0$ so that $\forall j \geq i \geq i'$

$\| \bar{u}_i^j - \bar{v}_i^j \|_{\bar{W}^{k+1}_{w_j}(K_{i_0}, h_j)}$

$\geq (1-\varepsilon) \| \bar{u}_i^j \|_{\bar{W}^{k+1}_{w_j}(K_{i_0}, h_j)}$

$V \subset W^*_w(S, g)$ linear closed

$\{\bar{V}^j_i\}_{i,j} \Longrightarrow V^{\perp}$: ∞遠で V に直交

も同様の定義.

Recall:

$$0 \to W^{k+2}_w(S,g) \xrightarrow{d_A} W^{k+1}_w(S,g) \xrightarrow{d_A^+} W^k_w(S,g) \to 0$$

フレドホルム.

これを用いることで.

Lemma. $\begin{cases} \bar{V}^j_i = \operatorname{im} d_{\bar{A}_j} \subset \bar{W}^{k+1}_{w_j}(K_i, h_j) \\ V = \operatorname{im} d_A \subset W^{k+1}_w(S, g) \end{cases}$

について. $\{\bar{V}^j_i\}_{i,j} \Longrightarrow V$ 収束
(∞遠で含まれていて直交成分なし.

☺ 全射性

$u = d_A f \in W^{k+1}_w(S,g)$, $\|u\|_{W^{k+1}_w} = 1$ をとる.

フレドホルム性より. $\exists\, C > 0$

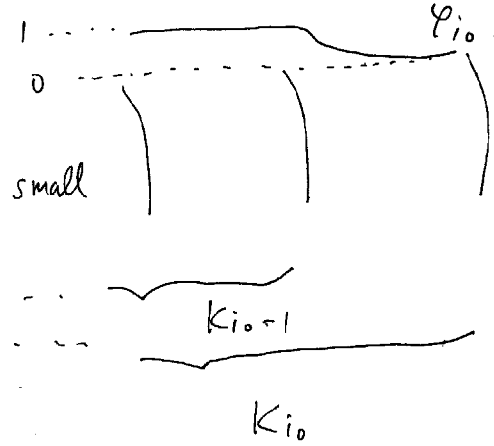
$$\|f\|_{W^{k+2}_w} \leq C \cdot \|d_A f\|_{W^{k+1}_w}.$$

$\forall\, \varepsilon > 0$ について. $\exists\, i_0$.

$\|f\|_{W^{k+2}_w(K_{i_0} \setminus K_{i_0-1})}$: small.

φ_{i_0} : cut off f.c.n.

$\|f - \varphi_{i_0} f\|_{W^{k+2}_w}$: small



$$\|u - d_A(\varphi_{i_0} f)\|_{W^{k+1}_w} < \varepsilon.$$

$\varphi_i f \in \bar{W}^{k+2}_{w_j}(K_i, h_j)$ より ⌟

(comparison method)

$U = \ker d_A^+ \supset V = \operatorname{im} d_A$,

Key Proposition

$$\begin{cases} \cdot\ \{\bar{s}^j_i\}_{i,j} \Longrightarrow U : \infty \text{ 遠で } U \text{ に含まれる} \\ \cdot\ \{\bar{s}^j_i\}_{i,j} \Longrightarrow V^\perp : \infty \text{ 遠で } V \text{ に直交} \end{cases}$$

$\Longrightarrow$ $\{\bar{s}^j_i\}_{i,j} \Longrightarrow T = U \cap V^\perp = H^1_A$

∞ 遠で T に含まれる

特に T は有限次元

系

$$\begin{cases} \cdot\ \{\bar{v}^j_i\}_{i,j} : \text{有界列} \longrightarrow V^\perp \text{ at } \infty \\ \cdot\ \lim_{i,j\to\infty} d^+_{\bar{A}_j}(\bar{v}^j_i) = 0 \end{cases}$$

$\Longrightarrow$ $\{\bar{v}^j_i\}_{i,j} \longrightarrow T$ at ∞.

Coulomb gauge-like condition

$\bar{\alpha}^j_i \in \overline{W}^{k+1}_{w_j}((K_i, h_j);\ \Lambda^1 \otimes \operatorname{ad} E)$

有界列

$\bar{A}'_j \equiv \bar{A}_j + \bar{\alpha}^j_i$

Proposition

$\exists\, c > 0$ so that $\forall \{\bar{\alpha}^j_i\}_{i,j}$, $\|\bar{\alpha}^j_i\| \leq c$

$\exists\, \bar{U}^j_i \in \overline{W}^{k+2}_{w_j}(K_i, h_j) \cap \operatorname{Aut}(E|K_i)$

such that after taking subindices of (i, j),

$$\begin{cases} \cdot\ \bar{\beta}^j_i = (\bar{U}^j_i)^*(\bar{A}'_j) - \bar{A}_j \\ \qquad \Longrightarrow V^\perp = (\operatorname{im} d_A)^\perp. \\ \cdot\ \|\bar{\beta}^j_i\| \leq {}^\exists C. \end{cases}$$

一様局所座標系の構成

Comparison method

$$F_s^+ : A_j + W_{w_j}^{k+1}(M, h_j) \times B \to W_{w_j}^{k}(M, h_j)$$

微分

$$D_{A_j}^+ : W_{w_j}^{k+1}(M, h_j) \times T_0 B \twoheadrightarrow W_{w_j}^{k}(M, h_j)$$

全射

同様に

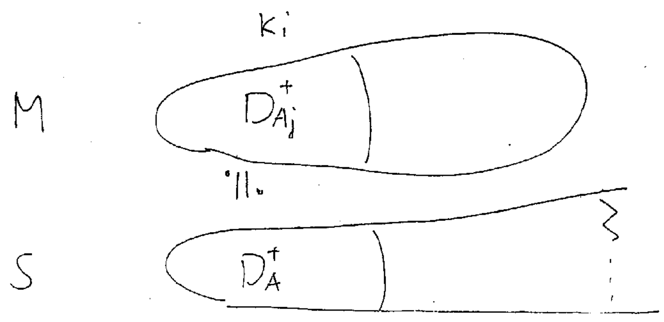
$$F_s^+ : A + W_w^{k+1}(S, g) \times B \to W_w^{k}(S, g)$$

微分

$$D_A^+ : W_w^{k+1}(S, g) \times T_0 B \twoheadrightarrow W_w^{k}(S, g)$$

全射

$A_j \to A$ without bubble



陰関数定理より

$$\begin{cases} \exists\, \varepsilon_j > 0, \quad M_1^j \times M_2^j \subset B_{\varepsilon_j}(0) \subset W_{w_j}^{k+1}(M, h_j) \times T_0 B \\ \exists\, G_j : M_1^j \to M_2^j \end{cases}$$

s.t. $(A', b) \in M_1^j \times M_2^j$, $F_b^+(A') = 0$, $(A', b) = (m_1, m_2)$

$\Leftrightarrow$ $m_2 = G_j(m_1)$.

一般に $\varepsilon_j \to 0 \quad (j \to \infty)$

$$\begin{cases} \exists\, \varepsilon > 0, \quad M_1 \times M_2 \subset B_\varepsilon(0) \subset W_w^{k+1}(S, g) \times T_0 B \\ \exists\, G : M_1 \to M_2 \end{cases}$$

s.t. $(A', b) = (m_1, m_2) \in M_1 \times M_2$ $F_b^+(A') = 0$

$\Leftrightarrow$ $m_2 = G(m_1)$.

$$\begin{cases} \exists\, \varepsilon > 0, \quad M_1 \times M_2 \subset B_\varepsilon(0) \subset W^{k+1}_w(S, g) \times T_0 \mathcal{B}. \\ \exists\, G : M_1 \to M_2 \end{cases}$$

s.t. $(A', b) = (m_1, m_2) \in M_1 \times M_2$

$$F^+_b(A') = 0$$

$$\Leftrightarrow \quad m_2 = G(m_1)$$

Rem. $\| D^+_A(u) \| \geq C \|u\|, \quad u \in (\operatorname{Ker} D^+_A)^\perp$

ε は C に depend.

$$\boxed{D^+_{\bar{A}} : \overline{W}^{k+1}_w(K_i, g) \times T_0 \mathcal{B} \twoheadrightarrow \overline{W}^k_w(K_i, g)}$$

$$\begin{cases} \overline{M}_1 \times \overline{M}_2 \subset \overline{B}_\varepsilon(0) \subset \overline{W}^{k+1}_w(K_i, g) \times T_0 \mathcal{B}. \\ \exists\, \overline{G} : \overline{M}_1 \to \overline{M}_2 \end{cases}$$

s.t. $(\bar{A}', b) = (\overline{m}_1, \overline{m}_2) \in \overline{M}_1 \times \overline{M}_2$

$$\overline{F}^+_b(\bar{A}') = 0$$

$$\Leftrightarrow \quad \overline{m}_2 = \overline{G}(m_1).$$

前 の key lemma.

Projections

$$pr : W^*_{w_j}(M, h_j) \longrightarrow \overline{W}^*_{w_j}(K_i, h_j)$$

$$pr : W^*_w(S, g) \longrightarrow \overline{W}^*_w(K_i, g)$$

Lemma. $\forall\, \varepsilon > 0, \ \exists\, N \gg 0$ s.t. $\forall j \geq i + N$.

$$(1 - \varepsilon) \|\cdot\|_{\overline{W}^*_{w_j}(K_i, h_j)} \leq \|\cdot\|_{\overline{W}^*_w(K_i, g)} \leq (1 + \varepsilon) \|\cdot\|_{\overline{W}^*_{w_j}(K_i, h_j)}$$

Compare :

$$D^+_{\bar{A}_j} : \overline{W}^{k+1}_{w_j}(K_i, h_j) \times T_0 \mathcal{B} \twoheadrightarrow \overline{W}^k_{w_j}(K_i, h_j)$$

$$D^+_{\bar{A}} : \overline{W}^{k+1}_w(K_i, g) \times T_0 \mathcal{B} \twoheadrightarrow \overline{W}^k_w(K_i, g)$$

系1. $j \geq i+N$ について、$\exists\, C>0$.

$$\| D^+_{\bar{A}_j}(\bar{u}) \|_{\bar{W}^k_{w_j}(K_i, h_j)} \geq C \| \bar{u} \|_{\bar{W}^{k+1}_{w_j}(K_i, h_j)}$$

$$\left(\bar{u} \in (\operatorname{Ker} D^+_{\bar{A}_j})^\perp \right).$$

ここで. C は. i, j に依らない

系2. $j \geq i+N$ について.

$\exists\, \varepsilon > 0$. $\bar{M}^j_1 \times \bar{M}^j_2 \subset B_\varepsilon(0) \subset \bar{W}^{k+1}_{w_j}(K_i, h_j) \times T_0 B$.

$\exists\, \bar{G}_j : \bar{M}^j_1 \longrightarrow \bar{M}^j_2$

s.t.

- $(\bar{A}', b) = (\bar{m}_1, \bar{m}_2) \in \bar{M}^j_1 \times \bar{M}^j_2$

  $\bar{F}^+_b(\bar{A}') = 0 \iff \bar{m}_2 = \bar{G}_j(\bar{m}_1)$

- $\| d\bar{G}_j \| \leq C$ (C は i, j に依らない)

Perturbations to normal direction of the moduli space at [A]

$U = \operatorname{ker} d_A^+ \supset V = \operatorname{im} d_A$,

$\cap$

$W^{k+1}_w(S, g : \Lambda^1 \otimes \operatorname{ad} E) \supset T = U \cap V^\perp$

$$\boxed{T = T_{[A]} M(A)}$$

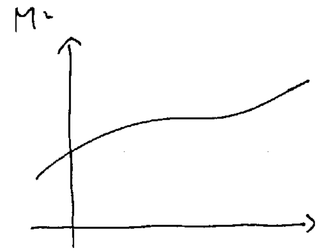
Recall ∞ dim implicit function theorem.

$$M_1 \times M_2 \subset B_\varepsilon(0) \subset W^{k+1}_w(S, g) \times T_0 B.$$

$(A', b) = (m_1, m_2) \in M_1 \times M_2$

$F^+_b(A') = 0$

$\iff m_2 = G(m_1)$



Graph(G) = Local Solutions.

$\cong$

G_r.

$$\begin{cases} D \subset T : \text{unit ball} \\ B_1 \subset T_0 B : \quad \text{〃} \end{cases}$$

Put $\underline{\widetilde{Gr}(D,0) \equiv \mathrm{Graph}(G) \cap (D \times B_1)}$

$pr : W_w^{k+1}(S, g) \times T_0 B \to T_0 B$
projection

Proposition

$\exists\, C > 0$, $\exists\, b_t \subset B$ smooth path
$(b_0 = 0)$

so that
$$\| b_t - pr \circ \widetilde{Gr}(D,0) \| \geq C t$$
$$(0 \leq t \leq 1)$$
holds

Final Step

仮定より $Q(E) \neq 0$.

$$\Rightarrow \begin{cases} \exists\, A_j^t ,\ (A_j^0 = A_j) \ \text{over } E \to (M, h_j) \\ F_s^+(A_j^t, b_t) = 0 \quad (0 \leq t \leq 1) \end{cases}$$

Recall $\underline{\text{一様局所座標系}}$

$$\begin{cases} \overline{M}_1^j \times \overline{M}_2^j \subset B_\varepsilon(0) \subset \overline{W}_{w_j}^{k+1}(K_i, h_j) \times T_0 B . \\ \overline{G}_j : \overline{M}_1^j \to \overline{M}_2^j \end{cases}$$

$(\bar{A}', b) = (\bar{m}_1, \bar{m}_2) \in \overline{M}_1^j \times \overline{M}_2^j$

$F_b^+(\bar{A}') = 0$

$\Leftrightarrow \quad \bar{m}_2 = \overline{G}_j(\bar{m}_1)$.

Proposition

$$\boxed{\begin{array}{l} \exists\, \bar{C}_j \in \bar{W}^{k+1}_{w_j}(K_i, h_j) \text{ such that} \\ (\bar{A}_j - \bar{C}_j,\ b) \in \bar{M}^j_1 \times \bar{M}^2_j \\ F^+_b(\bar{C}_j) = 0. \end{array}}$$

(∵) $\bar{A}^t_j = \bar{A}_j + \bar{\alpha}^t_j \in \bar{W}^{k+1}_{w_j}(K_i, h_j)$

Lemma. $\exists\, t_0 > 0$. s.t.

$(\bar{\alpha}^{t_0}_j, b_{t_0}) \in \bar{M}^j_1 \times \bar{M}^j_2 \subset B_\varepsilon(0)$

(∵) Suppose contrary

$\Longrightarrow \exists\, t_m \to 0$, $i_m, j_m \to \infty$, $m \to \infty$

$\| \bar{A}_j - \bar{A}^{t_m}_{j_m} \|_{\bar{W}^{k+1}_{w_{j_m}}(K_{i_m}, h_{j_m})} = \varepsilon.$

$b_{t_m} \to 0$ より. $\exists\, 0 \le s_m \le t_m$ について.

$$\bar{\alpha}_m \equiv \bar{A}_j - \bar{A}^{s_m}_{j_m}$$

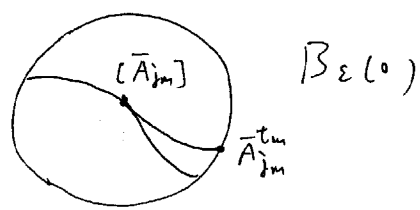
とおくと.

$$\{\bar{\alpha}_m\}. \Longrightarrow T = T_{[A]} M(A) \quad m \to \infty$$

(∵ Coulomb gauge-like condition)

$\therefore\ s_m^{-1} \| \bar{\alpha}_m - \operatorname{Ker} d^+_A \|_{\bar{W}^{k+1}_{w_{j_m}}(K_{i_m}, h_{j_m})} \to 0$

($\|d\bar{G}_j\| \le C\varepsilon$ 用いる)



$0 = \| b_{s_m} - \mathrm{pr}(\bar{\alpha}_m, b_{s_m}) \|$

$\ge \underbrace{\| b_{s_m} - \mathrm{pr}\,\widetilde{Gr}(D, 0) \|}_{\substack{\vee\!\shortparallel \\ C \cdot s_m}} - \underbrace{\| \mathrm{pr}(\bar{\alpha}_m, b_{s_m}) - \mathrm{pr}\,\widetilde{Gr}(D,0) \|}_{\substack{\le \varepsilon_m \cdot s_m \\ b_0}}$

$\ge \frac{C}{2} \cdot s_m.$ ムジュン」

以上から. $\exists\, t_0 > 0$

$$\| \overline{A}_j - \overline{A}_j^{t_0} \|_{W^{k+1}_{w_j}(K_i, g_j)} \leqq C.$$

$$\begin{array}{lccl} & \| \overline{A}_j & - & \overline{A}_j^{t_0} \| \\ i,j \to \infty & \downarrow & & \downarrow \quad \text{(Rellich)} \\ & A & & B \quad \text{over } (S, g). \end{array}$$

$$\Longrightarrow \begin{cases} \cdot \ \| A - B \|_{W^{k+1}_{w}(S, g)} \leqq C \\ \circ \ \ F^+(B, b_{t_0}) = 0. \end{cases}$$

$b = b_{t_0}$ ととり直して. 求める解 B を得た。